

# On the wellposedness of the KdV equation on the space of pseudomeasures

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October 4, 2016

## Abstract

In this paper we prove a wellposedness result of the KdV equation on the space of periodic pseudo-measures, also referred to as the Fourier Lebesgue space  $\mathcal{FL}^\infty(\mathbb{T}, \mathbb{R})$ , where  $\mathcal{FL}^\infty(\mathbb{T}, \mathbb{R})$  is endowed with the weak\* topology. Actually, it holds on any weighted Fourier Lebesgue space  $\mathcal{FL}^{s, \infty}(\mathbb{T}, \mathbb{R})$  with  $-1/2 < s \leq 0$  and improves on a wellposedness result of Bourgain for small Borel measures as initial data. A key ingredient of the proof is a characterization for a distribution  $q$  in the Sobolev space  $H^{-1}(\mathbb{T}, \mathbb{R})$  to be in  $\mathcal{FL}^\infty(\mathbb{T}, \mathbb{R})$  in terms of asymptotic behavior of spectral quantities of the Hill operator  $-\partial_x^2 + q$ . In addition, wellposedness results for the KdV equation on the Wiener algebra are proved.

**Keywords.** KdV equation, well-posedness, Birkhoff coordinates

**2000 AMS Subject Classification.** 37K10 (primary) 35Q53, 35D05 (secondary)

## 1 Introduction

In this paper we consider the initial value problem for the Korteweg-de Vries equation on the circle  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ ,

$$\partial_t u = -\partial_x^3 u + 6u\partial_x u, \quad x \in \mathbb{T}, \quad t \in \mathbb{R}. \quad (1)$$

Our goal is to improve the result of Bourgain [2] on global wellposedness for solutions evolving in the Fourier Lebesgue space  $\mathcal{FL}_0^\infty$  with small Borel measures as initial data. The space  $\mathcal{FL}_0^\infty$  consists of 1-periodic distributions  $q \in S'(\mathbb{T}, \mathbb{R})$

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\*Partially supported by the Swiss National Science Foundation

†Partially supported by the Swiss National Science Foundation

whose Fourier coefficients  $q_k = \langle q, e^{ik\pi x} \rangle$ ,  $k \in \mathbb{Z}$ , satisfy  $(q_k)_{k \in \mathbb{Z}} \in \ell^\infty(\mathbb{Z}, \mathbb{R})$  and  $q_0 = 0$ . Here and in the sequel we view for convenience 1-periodic distributions as 2-periodic ones and denote by  $\langle f, g \rangle$  the  $L^2$ -inner product  $\frac{1}{2} \int_0^2 f(x) \overline{g(x)} \, dx$  extended by duality to  $S'(\mathbb{R}/2\mathbb{Z}, \mathbb{C}) \times C^\infty(\mathbb{R}/2\mathbb{Z}, \mathbb{C})$ . We point out that  $q_{2k+1} = 0$  for any  $k \in \mathbb{Z}$  since  $q$  is a 1-periodic distribution. We succeed in dropping the smallness condition on the initial data and can allow for arbitrary initial data  $q \in \mathcal{F}\ell_0^\infty$ . In fact, our wellposedness results hold true for any of the spaces  $\mathcal{F}\ell_0^{s,\infty}$  with  $-1/2 < s \leq 0$  where

$$\mathcal{F}\ell_0^{s,\infty} = \{q \in S'(\mathbb{T}, \mathbb{R}) : q_0 = 0 \text{ and } \|(q_k)_{k \in \mathbb{Z}}\|_{s,\infty} < \infty\},$$

and

$$\|(q_k)_{k \in \mathbb{Z}}\|_{s,\infty} := \sup_{k \in \mathbb{Z}} \langle k \rangle^s |q_k|, \quad \langle \alpha \rangle := 1 + |\alpha|.$$

Informally stated, our result says that for any  $-1/2 < s \leq 0$ , the KdV equation is globally  $C^0$ -wellposed on  $\mathcal{F}\ell_0^{s,\infty}$ . To state it more precisely, we first need to recall the wellposedness results established in [17] on the Sobolev space  $H_0^{-1} \equiv H_0^{-1}(\mathbb{T}, \mathbb{R})$ . Let  $-\infty \leq a < b \leq \infty$  be given. A continuous curve  $\gamma: (a, b) \rightarrow H_0^{-1}$  with  $\gamma(0) = q \in H_0^{-1}$  is called a solution of (1) with initial data  $q$  if and only if for any sequence of  $C^\infty$ -potentials  $(q^{(m)})_{m \geq 1}$  converging to  $q$  in  $H_0^{-1}$ , the corresponding sequence  $(\mathcal{S}(t, q^{(m)}))_{m \geq 1}$  of solutions of (1) with initial data  $q^{(m)}$  converges to  $\gamma(t)$  in  $H_0^{-1}$  for any  $t \in (a, b)$ . In [17] it was proved that the KdV equation is globally in time  $C^0$ -wellposed meaning that for any  $q \in H_0^{-1}$  (1) admits a solution  $\gamma: \mathbb{R} \rightarrow H_0^{-1}$  with initial data in the above sense and for any  $T > 0$  the solution map  $\mathcal{S}: H_0^{-1} \rightarrow C([-T, T], H_0^{-1})$  is continuous. Note that for any  $-1/2 < s \leq 0$ ,  $\mathcal{F}\ell_0^{s,\infty}$  continuously embeds into  $H_0^{-1}$ . On  $\mathcal{F}\ell_0^{s,\infty}$  we denote by  $\tau_{w*}$  the weak\* topology  $\sigma(\mathcal{F}\ell_0^{s,\infty}, \mathcal{F}\ell_0^{-s,1})$ . We refer to Appendix B for a discussion.

**Theorem 1.1** *For any  $q \in \mathcal{F}\ell_0^{s,\infty}$  with  $-1/2 < s \leq 0$ , the solution curve  $t \mapsto \mathcal{S}(t, q)$  evolves in  $\mathcal{F}\ell_0^{s,\infty}$ . It is bounded,  $\sup_{t \in \mathbb{R}} \|\mathcal{S}(t, q)\|_{s,\infty} < \infty$  and continuous with respect to the weak\* topology  $\tau_{w*}$ .  $\times$*

*Remark 1.2.* It is easy to see that for generic initial data, the solution curve  $t \mapsto \mathcal{S}_{\text{Airy}}(t, q)$  of the Airy equation,  $\partial_t u = -\partial_x^3 u$ , is not continuous with respect to the norm topology of  $\mathcal{F}\ell_0^{s,\infty}$ . A similar result holds true for the KdV equation at least for small initial data – see Section 4.  $\rightarrow$

We say that a subset  $V \subset \mathcal{F}\ell_0^{s,\infty}$  is KdV-invariant if for any  $q \in \mathcal{F}\ell_0^{s,\infty}$ ,  $\mathcal{S}(t, q) \in V$  for any  $t \in \mathbb{R}$ .

**Theorem 1.3** *Let  $V \subset \mathcal{F}\ell_0^{s,\infty}$  with  $-1/2 < s \leq 0$  be a KdV-invariant  $\|\cdot\|_{s,\infty}$ -norm bounded subset, that is*

$$\mathcal{S}(t, q) \in V, \quad \forall t \in \mathbb{R}, q \in V; \quad \sup_{q \in V} \|q\|_{s,\infty} < \infty.$$

Then for any  $T > 0$ , the restriction of the solution map  $\mathcal{S}$  to  $V$  is weak\* continuous,

$$\mathcal{S}: (V, \tau_{w*}) \rightarrow C([-T, T], (V, \tau_{w*})). \quad \times$$

By the same methods we also prove that the KdV equation is globally  $C^\omega$ -wellposed on the Wiener algebra  $\mathcal{F}\ell_0^{0,1}$  – see Section 5 where such a result is proved for the weighted Fourier Lebesgue space  $\mathcal{F}\ell_0^{N,1}$ ,  $N \in \mathbb{Z}_{\geq 0}$ .

*Method of proof.* Theorem 1.1 and Theorem 1.3 are proved by the method of normal forms. We show that the restriction of the Birkhoff map  $\Phi: H_0^{-1} \rightarrow \ell_0^{-1/2,2}$   $q \mapsto z(q) = (z_n(q))_{n \in \mathbb{Z}}$ , constructed in [10], to  $\mathcal{F}\ell_0^{s,\infty}$  is a map with values in  $\ell_0^{s+1/2,\infty}(\mathbb{Z}, \mathbb{C})$ , having the following properties:

**Theorem 1.4** *For any  $-1/2 < s \leq 0$ ,  $\Phi: \mathcal{F}\ell_0^{s,\infty} \rightarrow \ell_0^{s,\infty}$  is a bijective, bounded, real analytic map between the two Banach spaces. Near the origin,  $\Phi$  is a local diffeomorphism. When restricted to any  $\|\cdot\|_{s,\infty}$ -norm bounded subset  $V \subset \mathcal{F}\ell_0^{s,\infty}$ ,  $\Phi: V \rightarrow \Phi(V)$  is a homeomorphism when  $V$  and  $\Phi(V)$  are endowed with the weak\* topologies  $\sigma(\mathcal{F}\ell_0^{s,\infty}, \mathcal{F}\ell_0^{-s,1})$  and  $\sigma(\ell_0^{s+1/2,\infty}, \ell_0^{-(s+1/2),1})$ , respectively. Furthermore for any  $q \in \mathcal{F}\ell_0^{s,\infty}$ , the set  $\text{Iso}(q)$  of elements  $\tilde{q} \in H_0^{-1}$  so that  $-\partial_x^2 + q$  and  $-\partial_x^2 + \tilde{q}$  have the same periodic spectrum is a  $\|\cdot\|_{s,\infty}$ -norm bounded subset of  $\mathcal{F}\ell_0^{s,\infty}$  and hence  $\Phi: \text{Iso}(q) \rightarrow \Phi(\text{Iso}(q))$  is a homeomorphism when  $\text{Iso}(q)$  and  $\Phi(\text{Iso}(q))$  are endowed with the weak\* topologies.*  $\times$

*Remark 1.5.* Note that by [10] for any  $q \in H_0^{-1}$ ,  $\Phi(\text{Iso}(q)) = \mathcal{T}_{\Phi(q)}$  where

$$\mathcal{T}_{\Phi(q)} = \{\tilde{z} = (\tilde{z}_k)_{k \in \mathbb{Z}} \in \ell_0^{-1/2,2} : |\tilde{z}_k| = |\Phi(q)_k| \ \forall k \in \mathbb{Z}\}. \quad (2)$$

Furthermore, since by Theorem 1.4 for any  $q \in \mathcal{F}\ell_0^{s,\infty}$ ,  $-1/2 < s \leq 0$ ,  $\text{Iso}(q)$  is bounded in  $\mathcal{F}\ell_0^{s,\infty}$ , the weak\* topology on  $\text{Iso}(q)$  coincides with the one induced by the norm  $\|\cdot\|_{\sigma,p}$  for any  $-1/2 < \sigma < s$ ,  $2 \leq p < \infty$  with  $(s - \sigma)p > 1$  – cf. Lemma B.1 from Appendix B.  $\rightarrow$

Key ingredient for studying the restriction of the Birkhoff map to the Banach spaces  $\mathcal{F}\ell_0^{s,\infty}$  are pertinent asymptotic estimates of spectral quantities of the Schrödinger operator  $-\partial_x^2 + q$ , which appear in the estimates of the Birkhoff coordinates in [10, 7] – see Section 2. The proofs of Theorem 1.1 and Theorem 1.3 then are obtained by studying the restriction of the solution map  $\mathcal{S}$ , defined in [17] on  $H_0^{-1}$ , to  $\mathcal{F}\ell_0^{s,\infty}$ . To this end, the KdV equation is expressed in Birkhoff coordinates  $z = (z_n)_{n \in \mathbb{Z}}$ . It takes the form

$$\partial_t z_n = -i\omega_n z_n, \quad \partial_t z_{-n} = i\omega_n z_{-n}, \quad n \geq 1,$$

where  $\omega_n$ ,  $n \geq 1$ , are the KdV frequencies. For  $q \in H_0^1$ , these frequencies are defined in terms of the KdV Hamiltonian  $H(q) = \int_0^1 (\frac{1}{2}(\partial_x q)^2 + q^3) dx$ . When

viewed as a function of the Birkhoff coordinates,  $H$  is a real analytic function of the actions  $I_n = z_n z_{-n}$ ,  $n \geq 1$ , alone and  $\omega_n$  is given by

$$\omega_n = \partial_{I_n} H.$$

For  $q \in H_0^{-1}$ , the KdV frequencies are defined by analytic extension – see [11] for novel formulas allowing to derive asymptotic estimates.

*Related results.* The wellposedness of the KdV equation on  $\mathbb{T}$  has been extensively studied - c.f. e.g. [20] for an account on the many results obtained so far. In particular, based on [1] and [18] it was proved in [3] that the KdV equation is globally uniformly  $C^0$ -wellposed and  $C^\omega$ -wellposed on the Sobolev spaces  $H_0^s(\mathbb{T}, \mathbb{R})$  for any  $s \geq -1/2$ . In [17] it was shown that the KdV equation is globally  $C^0$ -wellposed in the Sobolev spaces  $H_0^s$ ,  $-1 \leq s < 1/2$  and in [11] it was proved that for  $-1 < s < -1/2$  and  $T > 0$ , the solution map  $H_0^s \rightarrow C([-T, T], H_0^s)$  is nowhere locally uniformly continuous. In [20], it was shown that the KdV equation is illposed in  $H_0^s$  for  $s < -1$ . Most closely related to Theorem 1.1 and Theorem 1.3 are the wellposedness results of Bourgain [2] for initial data given by Borel measures which we have already discussed at the beginning of the introduction and the recent wellposedness results in [7] on the Fourier Lebesgue spaces  $\mathcal{F}\ell_0^{s,p}$  for  $-1/2 \leq s \leq 0$  and  $2 \leq p < \infty$ .

*Notation.* We collect a few notations used throughout the paper. For any  $s \in \mathbb{R}$  and  $1 \leq p \leq \infty$ , denote by  $\ell_{0,\mathbb{C}}^{s,p}$  the  $\mathbb{C}$ -Banach space of complex valued sequences given by

$$\ell_{0,\mathbb{C}}^{s,p} := \{z = (z_k)_{k \in \mathbb{Z}} \subset \mathbb{C} : z_0 = 0; \quad \|z\|_{s,p} < \infty\},$$

where

$$\|z\|_{s,p} := \left( \sum_{k \in \mathbb{Z}} \langle n \rangle^{sp} |z_n|^p \right)^{1/p}, \quad 1 \leq p < \infty, \quad \|z\|_{s,\infty} := \sup_{k \in \mathbb{Z}} \langle n \rangle^s |z_n|,$$

and by  $\ell_0^{s,p}$  the real subspace

$$\ell_0^{s,p} := \{z = (z_k)_{k \in \mathbb{Z}} \in \ell_{0,\mathbb{C}}^{s,p} : z_{-k} = \overline{z_k} \ \forall \ k \geq 1\}.$$

By  $\ell_{\mathbb{R}}^{s,p}$  we denote the  $\mathbb{R}$ -subspace of  $\ell_0^{s,p}$  consisting of real valued sequences  $z = (z_k)_{k \in \mathbb{Z}}$  in  $\mathbb{R}$ . Further, we denote by  $\mathcal{F}\ell_{0,\mathbb{C}}^{s,p}$  the Fourier Lebesgue space, introduced by Hörmander,

$$\mathcal{F}\ell_{0,\mathbb{C}}^{s,p} := \{q \in S'_{\mathbb{C}}(\mathbb{T}) : (q_k)_{k \in \mathbb{Z}} \subset \ell_{0,\mathbb{C}}^{s,p}\}$$

where  $q_k$ ,  $k \in \mathbb{Z}$ , denote the Fourier coefficients of the 1-periodic distribution  $q$ ,  $q = \langle q, e_k \rangle$ ,  $e_k(x) := e^{ik\pi x}$ , and  $\langle \cdot, \cdot \rangle$  denotes the  $L^2$ -inner product,  $\langle f, g \rangle = \frac{1}{2} \int_0^2 f(x) \overline{g(x)} \, dx$ , extended by duality to a sesquilinear form on  $S'_{\mathbb{C}}(\mathbb{R}/2\mathbb{Z}) \times C_{\mathbb{C}}^\infty(\mathbb{R}/2\mathbb{Z})$ . Correspondingly, we denote by  $\mathcal{F}\ell_0^{s,p}$  the real subspace of  $\mathcal{F}\ell_{0,\mathbb{C}}^{s,p}$ ,

$$\mathcal{F}\ell_0^{s,p} := \{q \in S'_{\mathbb{C}}(\mathbb{T}) : (q_k)_{k \in \mathbb{Z}} \subset \ell_0^{s,p}\}.$$

In case  $p = 2$ , we also write  $H_0^s [H_{0,\mathbb{C}}^s]$  instead of  $\mathcal{F}\ell_0^{s,2} [\mathcal{F}\ell_{0,\mathbb{C}}^{s,2}]$  and refer to it as Sobolev space. Similarly, for the sequence spaces  $\ell_0^{s,2}$  and  $\ell_{0,\mathbb{C}}^{s,2}$  we sometimes write  $h_0^s [h_{0,\mathbb{C}}^s]$ . Occasionally, we will need to consider the sequence spaces  $\ell_{\mathbb{C}}^{s,p}(\mathbb{N}) \equiv \ell^{s,p}(\mathbb{N}, \mathbb{C})$  and  $\ell_{\mathbb{R}}^{s,p}(\mathbb{N}) \equiv \ell^{s,p}(\mathbb{N}, \mathbb{R})$  defined in an obvious way.

Note that for any  $z \in \ell_0^{s,p}$ ,  $I_k := z_k z_{-k} \geq 0$  for all  $k \geq 1$ . We denote by  $\mathcal{T}_z$  the torus given by

$$\mathcal{T}_z := \{\tilde{z} = (\tilde{z}_k)_{k \in \mathbb{Z}} \in \ell_0^{s,p} : \tilde{z}_k \tilde{z}_{-k} = z_k z_{-k}, \quad k \geq 1\}.$$

For  $1 \leq p < \infty$ ,  $\mathcal{T}_z$  is compact in  $\ell_0^{s,p}$  for any  $z \in \ell_0^{s,p}$  but for  $p = \infty$ , it is not compact in  $\ell^{s,\infty}$  for generic  $z$ . For any  $s \in \mathbb{R}$  and  $1 \leq p < \infty$ , the dual of  $\ell_0^{s,p}$  is given by  $\ell_0^{-s,p'}$  where  $p'$  is the conjugate of  $p$ , given by  $1/p + 1/p' = 1$ . In case  $p = 1$  we set  $p' = \infty$  and in case  $p = \infty$  we set  $p' = 1$ . We denote by  $\tau_{w*}$  the weak\* topology on  $\ell_0^{s,\infty}$  and refer to Appendix B for a discussion of the properties of  $\tau_{w*}$ .

## 2 Spectral theory

In this section we consider the Schrödinger operator

$$L(q) = -\partial_x^2 + q, \tag{3}$$

which appears in the Lax pair formulation of the KdV equation. Our aim is to relate the regularity of the potential  $q$  to the asymptotic behavior of certain spectral data.

Let  $q$  be a *complex potential* in  $H_{0,\mathbb{C}}^{-1} := H_0^{-1}(\mathbb{R}/\mathbb{Z}, \mathbb{C})$ . In order to treat periodic and antiperiodic boundary conditions at the same time, we consider the differential operator  $L(q) = -\partial_x^2 + q$ , on  $H^{-1}(\mathbb{R}/2\mathbb{Z}, \mathbb{C})$  with domain of definition  $H^1(\mathbb{R}/2\mathbb{Z}, \mathbb{C})$ . See Appendix C for a more detailed discussion. The spectral theory of  $L(q)$ , while classical for  $q \in L_{0,\mathbb{C}}^2$ , has been only fairly recently extended to the case  $q \in H_{0,\mathbb{C}}^{-1}$  – see e.g. [5, 9, 16, 19, 22, 7] and the references therein. The spectrum of  $L(q)$ , called the *periodic spectrum of  $q$*  and denoted by  $\text{spec } L(q)$ , is discrete and the eigenvalues, when counted with their multiplicities and ordered lexicographically – first by their real part and second by their imaginary part – satisfy

$$\lambda_0^+(q) \leq \lambda_1^-(q) \leq \lambda_1^+(q) \leq \dots, \quad \lambda_n^\pm(q) = n^2 \pi^2 + n \ell_n^2. \tag{4}$$

Furthermore, we define the *gap lengths*  $\gamma_n(q)$  and the *mid points*  $\tau_n(q)$  by

$$\gamma_n(q) := \lambda_n^+(q) - \lambda_n^-(q), \quad \tau_n(q) := \frac{\lambda_n^+(q) + \lambda_n^-(q)}{2}, \quad n \geq 1. \tag{5}$$

For  $q \in H_{0,\mathbb{C}}^{-1}$  we also consider the operator  $L_{\text{dir}}(q)$  defined as the operator  $-\partial_x^2 + q$  on  $H_{\text{dir}}^{-1}([0, 1], \mathbb{C})$  with domain of definition  $H_{\text{dir}}^1([0, 1], \mathbb{C})$ . See Appendix C

as well as [5, 9, 16, 19, 22] for a more detailed discussion. The spectrum of  $L_{\text{dir}}(q)$  is called the *Dirichlet spectrum of  $q$* . It is also discrete and given by a sequence of eigenvalues  $(\mu_n)_{n \geq 1}$ , counted with multiplicities, which when ordered lexicographically satisfies

$$\mu_1 \preceq \mu_2 \preceq \mu_2 \preceq \cdots, \quad \mu_n = n^2 \pi^2 + n \ell_n^2. \quad (6)$$

For our purposes we need to characterize the regularity of potentials  $q$  in *weighted Fourier Lebesgue spaces* in terms of the asymptotic behavior of certain spectral quantities. A normalized, symmetric, monotone, and submultiplicative *weight* is a function  $w: \mathbb{Z} \rightarrow \mathbb{R}$ ,  $n \mapsto w_n$ , satisfying

$$w_n \geq 1, \quad w_{-n} = w_n, \quad w_{|n|} \leq w_{|n|+1}, \quad w_{n+m} \leq w_n w_m,$$

for all  $n, m \in \mathbb{Z}$ . The class of all such weights is denoted by  $\mathcal{M}$ . For  $w \in \mathcal{M}$ ,  $s \in \mathbb{R}$ , and  $1 \leq p \leq \infty$ , denote by  $\mathcal{F}\ell_{0,\mathbb{C}}^{w,s,p}$  the subspace of  $\mathcal{F}\ell_{0,\mathbb{C}}^{s,p}$  of distributions  $f$  whose Fourier coefficients  $(f_n)_{n \in \mathbb{Z}}$  are in the space  $\ell_{0,\mathbb{C}}^{w,s,p} = \{z = (z_n)_{n \in \mathbb{Z}} \in \ell_{0,\mathbb{C}}^{s,p} : \|z\|_{w,s,p} < \infty\}$  where for  $1 \leq p < \infty$

$$\|f\|_{w,s,p} := \|(f_n)_{n \in \mathbb{Z}}\|_{w,s,p} = \left( \sum_{n \in \mathbb{Z}} w_n^p \langle n \rangle^{sp} |f_n|^p \right)^{1/p}, \quad \langle \alpha \rangle := 1 + |\alpha|,$$

and for  $p = \infty$ ,

$$\|f\|_{w,s,\infty} := \|(f_n)_{n \in \mathbb{Z}}\|_{w,s,\infty} = \sup_{n \in \mathbb{Z}} w_n \langle n \rangle^s |f_n|.$$

To simplify notation, we denote the trivial weight  $w_n \equiv 1$  by  $\mathbf{o}$  and write  $\mathcal{F}\ell_{0,\mathbb{C}}^{s,p} \equiv \mathcal{F}\ell_{0,\mathbb{C}}^{\mathbf{o},s,p}$ .

As a consequence of (4)–(6) it follows that for any  $q \in H_{0,\mathbb{C}}^{-1}$ , the sequence of gap lengths  $(\gamma_n(q))_{n \geq 1}$  and the sequence  $(\tau_n(q) - \mu_n(q))_{n \geq 1}$  are both in  $\ell_{\mathbb{C}}^{-1,2}(\mathbb{N})$ . For  $q \in \mathcal{F}\ell_{0,\mathbb{C}}^{w,s,\infty}$ ,  $-1/2 < s \leq 0$ , the sequences have a stronger decay. More precisely, the following results hold:

**Theorem 2.1** *Let  $w \in \mathcal{M}$  and  $-1/2 < s \leq 0$ .*

(i) *For any  $q \in \mathcal{F}\ell_{0,\mathbb{C}}^{w,s,\infty}$ , one has  $(\gamma_n(q))_{n \geq 1} \in \ell_{\mathbb{C}}^{w,s,\infty}(\mathbb{N})$  and the map*

$$\mathcal{F}\ell_{0,\mathbb{C}}^{w,s,\infty} \rightarrow \ell_{\mathbb{C}}^{w,s,\infty}(\mathbb{N}), \quad q \mapsto (\gamma_n(q))_{n \geq 1},$$

*is locally bounded.*

(ii) *For any  $q \in \mathcal{F}\ell_{0,\mathbb{C}}^{w,s,\infty}$ , one has  $(\tau_n - \mu_n(q))_{n \geq 1} \in \ell_{\mathbb{C}}^{w,s,\infty}(\mathbb{N})$  and the map*

$$\mathcal{F}\ell_{0,\mathbb{C}}^{w,s,\infty} \rightarrow \ell_{\mathbb{C}}^{w,s,\infty}(\mathbb{N}), \quad q \mapsto (\tau_n(q) - \mu_n(q))_{n \geq 1},$$

*is locally bounded.*     $\times$

A key ingredient for studying the restriction of the Birkhoff map of the KdV equation, defined on  $H_0^{-1}$ , to  $\mathcal{F}\ell_0^{s,\infty}$  is the following spectral characterization for a potential  $q \in H_0^{-1}$  to be in  $\mathcal{F}\ell_0^{s,\infty}$ .

**Theorem 2.2** *Let  $q \in H_0^{-1}$  with gap lengths  $\gamma(q) \in \ell_{\mathbb{R}}^{s,\infty}$  for some  $-1/2 < s \leq 0$ . Then the following holds:*

- (i)  $q \in \mathcal{F}\ell_0^{s,\infty}$ .
- (ii)  $\text{Iso}(q) \subset \mathcal{F}\ell_0^{s,\infty}$ .
- (iii)  $\text{Iso}(q)$  is weak\* compact.  $\times$

*Remark 2.3.* For any  $-1/2 < s \leq 0$ , there are potentials  $q \in \mathcal{F}\ell^{s,\infty}$  so that  $\text{Iso}(q)$  is not compact in  $\mathcal{F}\ell^{s,\infty}$  – see item (iii) in Lemma 3.5.  $\rightarrow$

In the remainder of this section we prove Theorem 2.1 and Theorem 2.2 by extending the methods, used in [8, 21, 4] for potentials  $q \in L^2$ , for singular potentials. We point out that the spectral theory is only developed as far as needed.

## 2.1 Setup

We extend the  $L^2$ -inner product  $\langle f, g \rangle = \frac{1}{2} \int_0^2 f(x) \overline{g(x)} dx$  on  $L_{\mathbb{C}}^2(\mathbb{R}/2\mathbb{Z}) \equiv L^2(\mathbb{R}/2\mathbb{Z}, \mathbb{C})$  by duality to  $\mathcal{S}'_{\mathbb{C}}(\mathbb{R}/2\mathbb{Z}) \times C_{\mathbb{C}}^{\infty}(\mathbb{R}/2\mathbb{Z})$ . Let  $e_n(x) = e^{i\pi n x}$ ,  $n \in \mathbb{Z}$ , and for  $w \in \mathcal{M}$ ,  $s \in \mathbb{R}$ , and  $1 \leq p \leq \infty$  denote by  $\mathcal{F}\ell_{\star, \mathbb{C}}^{w,s,p}$  the space of 2-periodic, complex valued distributions  $f \in \mathcal{S}'_{\mathbb{C}}(\mathbb{R}/2\mathbb{Z})$  so that the sequence of their Fourier coefficients  $f_n = \langle f, e_n \rangle$  is in the space  $\ell_{\mathbb{C}}^{w,s,p} = \{z = (z_n)_{n \in \mathbb{Z}} \subset \mathbb{C} : \|z\|_{w,s,p} < \infty\}$ . To simplify notation, we write  $\mathcal{F}\ell_{\star, \mathbb{C}}^{s,p} \equiv \mathcal{F}\ell_{\star, \mathbb{C}}^{0,s,p}$ .

In the sequel we will identify a potential  $q \in \mathcal{F}\ell_{0, \mathbb{C}}^{w,s,\infty}$  with the corresponding element  $\sum_{n \in \mathbb{Z}} q_n e_n$  in  $\mathcal{F}\ell_{\star, \mathbb{C}}^{w,s,\infty}$  where  $q_n$  is the  $n$ th Fourier coefficient of the potential obtained from  $q$  by viewing it as a distribution on  $\mathbb{R}/2\mathbb{Z}$  instead of  $\mathbb{R}/\mathbb{Z}$ , i.e.,  $q_{2n} = \langle q, e_{2n} \rangle$ , whereas  $q_{2n+1} = \langle q, e_{2n+1} \rangle = 0$  and  $q_0 = \langle q, 1 \rangle = 0$ . We denote by  $V$  the operator of multiplication by  $q$  with domain  $H_{\mathbb{C}}^1(\mathbb{R}/2\mathbb{Z})$ . See Appendix C for a detailed discussion of this operator as well as the operator  $L(q)$  introduced in (3). When expressed in its Fourier series, the image  $Vf$  of  $f = \sum_{n \in \mathbb{Z}} f_n e_n \in H_{\mathbb{C}}^1(\mathbb{R}/2\mathbb{Z})$  is the distribution  $Vf = \sum_{n \in \mathbb{Z}} (\sum_{m \in \mathbb{Z}} q_{n-m} f_m) e_n \in H_{\mathbb{C}}^{-1}(\mathbb{R}/2\mathbb{Z})$ . To prove the asymptotic estimates of the gap lengths stated in Theorem 2.1 we need to study the eigenvalue equation  $L(q)f = \lambda f$  for sufficiently large periodic eigenvalues  $\lambda$ . For  $q \in H_{0, \mathbb{C}}^{-1}$ , the domain of  $L(q)$  is  $H_{\mathbb{C}}^1(\mathbb{R}/2\mathbb{Z})$  and hence the eigenfunction  $f$  is an element of this space. It is shown in Appendix C that for  $q \in \mathcal{F}\ell_{0, \mathbb{C}}^{s,\infty}$  with  $-1/2 < s \leq 0$  and  $2 \leq p \leq \infty$ , one has  $f \in \mathcal{F}\ell_{\star, \mathbb{C}}^{s+2,p}$  and  $\partial_x^2 f, Vf \in \mathcal{F}\ell_{\star, \mathbb{C}}^{s,\infty}$ . Note that for  $q = 0$  and any  $n \geq 1$ ,  $\lambda_n^+(0) = \lambda_n^-(0) = n^2 \pi^2$ , and the eigenspace

corresponding to the double eigenvalue  $\lambda_n^+(0) = \lambda_n^-(0)$  is spanned by  $e_n$  and  $e_{-n}$ . Viewing  $L(q) - \lambda_n^\pm(q)$  for  $n$  large as a perturbation of  $L(0) - \lambda_n^\pm(0)$ , we are led to decompose  $\mathcal{F}\ell_{\star, \mathbb{C}}^{s, \infty}$  into the direct sum  $\mathcal{F}\ell_{\star, \mathbb{C}}^{s, \infty} = \mathcal{P}_n \oplus \mathcal{Q}_n$ ,

$$\mathcal{P}_n = \text{span}\{e_n, e_{-n}\}, \quad \mathcal{Q}_n = \overline{\text{span}}\{e_k : k \neq \pm n\}. \quad (7)$$

The  $L^2$ -orthogonal projections onto  $\mathcal{P}_n$  and  $\mathcal{Q}_n$  are denoted by  $P_n$  and  $Q_n$ , respectively. It is convenient to write the eigenvalue equation  $Lf = \lambda f$  in the form  $A_\lambda f = Vf$ , where  $A_\lambda f = \partial_x^2 f + \lambda f$  and  $V$  denotes the operator of multiplication with  $q$ . Since  $A_\lambda$  is a Fourier multiplier, we write  $f = u + v$ , where  $u = P_n f$  and  $v = Q_n f$ , and decompose the equation  $A_\lambda f = Vf$  into the two equations

$$A_\lambda u = P_n V(u + v), \quad A_\lambda v = Q_n V(u + v), \quad (8)$$

referred to as  $P$ - and  $Q$ -equation. Since  $q \in H_{0, \mathbb{C}}^{-1}$ , it follows from [16] that  $\lambda_n^\pm(q) = n^2\pi^2 + n\ell_n^2$ . Hence for  $n$  sufficiently large,  $\lambda_n^\pm(q) \in S_n$  where  $S_n$  denotes the closed vertical strip

$$S_n := \{\lambda \in \mathbb{C} : |\Re \lambda - n^2\pi^2| \leq 12n\}, \quad n \geq 1. \quad (9)$$

Note that  $\{\lambda \in \mathbb{C} : \Re \lambda \geq 0\} \subset \bigcup_{n \geq 1} S_n$ . Given any  $n \geq 1$ ,  $u \in \mathcal{P}_n$ , and  $\lambda \in S_n$ , we derive in a first step from the  $Q$ -equation an equation for  $Vv$  which for  $n$  sufficiently large can be solved as a function of  $u$  and  $\lambda$ . In a second step, for  $\lambda$  a periodic eigenvalue in  $S_n$ , we solve the  $P$  equation for  $u$  after having substituted in it the expression of  $Vv$ . The solution of the  $Q$ -equation is then easily determined. Towards the first step note that for any  $\lambda \in S_n$ ,  $A_\lambda : \mathcal{Q}_n \cap \mathcal{F}\ell_{\star, \mathbb{C}}^{s+2, p} \rightarrow \mathcal{Q}_n$  is boundedly invertible as for any  $k \neq n$ ,

$$\min_{\lambda \in S_n} |\lambda - k^2\pi^2| \geq \min_{\lambda \in S_n} |\Re \lambda - k^2\pi^2| \geq |n^2 - k^2| \geq 1. \quad (10)$$

In order to derive from the  $Q$ -equation an equation for  $Vv$ , we apply to it the operator  $VA_\lambda^{-1}$  to get

$$Vv = VA_\lambda^{-1}Q_n V(u + v) = T_n V(u + v),$$

where

$$T_n \equiv T_n(\lambda) := VA_\lambda^{-1}Q_n : \mathcal{F}\ell_{\star, \mathbb{C}}^{w, s, \infty} \rightarrow \mathcal{F}\ell_{\star, \mathbb{C}}^{w, s, \infty}.$$

It leads to the following equation for  $\check{v} := Vv$

$$(\text{Id} - T_n(\lambda))\check{v} = T_n(\lambda)Vu. \quad (11)$$

To show that  $\text{Id} - T_n(\lambda)$  is invertible, we introduce for any  $s \in \mathbb{R}$ ,  $w \in \mathbb{M}$ , and  $l \in \mathbb{Z}$  the shifted norm of  $f \in \mathcal{F}\ell_{\star, \mathbb{C}}^{w, s, \infty}$ ,

$$\|f\|_{w, s, \infty; l} := \|fe_l\|_{w, s, \infty} = \left\| (w_{k+l} \langle k+l \rangle^s f_k)_{k \in \mathbb{Z}} \right\|_{\ell^p},$$



and denote by  $\|T_n\|_{w,s,\infty;l}$  the operator norm of  $T_n$  viewed as an operator on  $\mathcal{F}_{\star,\mathbb{C}}^{w,s,\infty}$  with norm  $\|\cdot\|_{w,s,\infty;l}$ . Furthermore, we denote by  $R_N f$ ,  $N \geq 1$ , the tail of the Fourier series of  $f \in \mathcal{F}_{\star,\mathbb{C}}^{w,s,\infty}$ ,

$$R_N f = \sum_{|k| \geq N} f_k e_k.$$

**Lemma 2.4** *Let  $-1/2 < s \leq 0$ ,  $w \in \mathcal{M}$ , and  $n \geq 1$  be given. For any  $q \in \mathcal{F}_{0,\mathbb{C}}^{w,s,\infty}$  and  $\lambda \in S_n$ ,*

$$T_n(\lambda): \mathcal{F}_{\star,\mathbb{C}}^{w,s,\infty} \rightarrow \mathcal{F}_{\star,\mathbb{C}}^{w,s,\infty}$$

*is a bounded linear operator satisfying the estimate*

$$\|T_n(\lambda)\|_{w,s,\infty;\pm n} \leq \frac{c_s}{n^{1/2-|s|}} \|q\|_{w,s,\infty}, \quad (12)$$

where  $c_s \geq 1$  is a constant depending only on  $s$  and decreasing monotonically in  $s$ . In particular,  $c_s$  does not depend on  $q$  nor on the weight  $w$ .  $\times$

*Proof.* Let  $s$  and  $w$  be given as in the statement of the lemma. Note that  $A_\lambda^{-1}: \mathcal{F}_{\star,\mathbb{C}}^{s,\infty} \rightarrow \mathcal{F}_{\star,\mathbb{C}}^{s+2,\infty}$  is bounded for any  $\lambda \in S_n$  and hence for any  $f \in \mathcal{F}_{\star,\mathbb{C}}^{s,\infty}$ ,  $q \in \mathcal{F}_{0,\mathbb{C}}^{s,\infty}$ , and  $\lambda \in S_n$ , the multiplication of  $A_\lambda^{-1} Q_n f$  with  $q$ , defined by

$$V A_\lambda^{-1} Q_n f = \sum_{m \in \mathbb{Z}} \left( \sum_{|k| \neq n} \frac{q_{m-k} f_k}{\lambda - k^2 \pi^2} \right) e_m \quad (13)$$

is a distribution in  $S'_\mathbb{C}(\mathbb{R}/2\mathbb{Z})$ . Note that  $T_n(\lambda)f = V A_\lambda^{-1} Q_n f$  and that its norm  $\|T_n(\lambda)f\|_{w,s,\infty;n}$  satisfies for any  $\lambda \in S_n$ ,

$$\|T_n f\|_{w,s,\infty;n} \leq \sup_{m \in \mathbb{Z}} \sum_{|k| \neq n} \frac{w_{m+n} \langle k+n \rangle^{|s|} \langle m-k \rangle^{|s|}}{|n+k| |n-k| \langle m+n \rangle^{|s|}} \frac{|q_{m-k}|}{\langle m-k \rangle^{|s|}} \frac{|f_k|}{\langle k+n \rangle^{|s|}},$$

where we have used (10). Since  $\langle m-k \rangle \leq \langle m+n \rangle \langle n+k \rangle$ ,  $-1/2 < s \leq 0$ , and  $\langle \nu \rangle / |\nu| \leq 2$ , we conclude

$$\frac{\langle k+n \rangle^{|s|} \langle m-k \rangle^{|s|}}{|n+k| |n-k| \langle m+n \rangle^{|s|}} \leq \frac{\langle k+n \rangle^{2|s|}}{|n+k| |n-k|} \leq \frac{2}{|n+k|^{1-2|s|} |n-k|}.$$

Hölder's inequality together with the submultiplicativity of the weight  $w$  then yields

$$\begin{aligned} \|T_n f\|_{w,s,\infty;n} &\leq 2 \sup_{m \in \mathbb{Z}} \sum_{|k| \neq n} \frac{1}{|n+k|^{1-2|s|} |n-k|} \frac{w_{m-k} |q_{m-k}|}{\langle m-k \rangle^{|s|}} \frac{w_{k+n} |f_k|}{\langle k+n \rangle^{|s|}} \\ &\leq 2 \left( \sum_{|k| \neq n} \frac{1}{|n+k|^{(1-2|s|)} |n-k|} \right) \|q\|_{w,s,\infty} \|f\|_{w,s,\infty;n}. \end{aligned}$$

One checks that

$$\sum_{|k| \neq n} \frac{1}{|n+k|^{(1-2|s|)}|n-k|} \leq \frac{c_s}{n^{1/2-|s|}},$$

Going through the arguments of the proof one sees that the same kind of estimates also lead to the claimed bound for  $\|T_n f\|_{w,s,\infty;-n}$ .  $\blacksquare$

Lemma 2.4 can be used to solve, for  $n$  sufficiently large, the equation (11) as well as the  $Q$ -equation (8) in terms of any given  $u \in \mathcal{P}_n$  and  $\lambda \in S_n$ .

**Corollary 2.5** *For any  $q \in \mathcal{F}_{0,\mathbb{C}}^{w,s,\infty}$  with  $-1/2 < s \leq 0$  and  $w \in \mathcal{M}$ , there exists  $n_s = n_s(q) \geq 1$  so that,*

$$2c_s \|q\|_{w,s,\infty} \leq n_s^{1/2-|s|}, \quad (14)$$

with  $c_s \geq 1$  the constant in (12) implying that for any  $\lambda \in S_n$ ,  $T_n(\lambda)$  is a  $1/2$  contraction on  $\mathcal{F}_{\star,\mathbb{C}}^{w,s,\infty}$  with respect to the norms shifted by  $\pm n$ ,  $\|T_n(\lambda)\|_{w,s,\infty;\pm n} \leq 1/2$ . The threshold  $n_s(q)$  can be chosen uniformly in  $q$  on bounded subsets of  $\mathcal{F}_{0,\mathbb{C}}^{w,s,\infty}$ . As a consequence, for  $n \geq n_s(q)$ , equation (11) and (8) can be uniquely solved for any given  $u \in \mathcal{P}_n$ ,  $\lambda \in S_n$ ,

$$\check{v}_{u,\lambda} = K_n(\lambda)T_n(\lambda)Vu \in \mathcal{F}_{\star,\mathbb{C}}^{s,\infty}, \quad K_n \equiv K_n(\lambda) := (\text{Id} - T_n(\lambda))^{-1}, \quad (15)$$

$$v_{u,\lambda} = A_\lambda^{-1}Q_nVu + A_\lambda^{-1}Q_n\check{v}_{u,\lambda} = A_\lambda^{-1}Q_nK_nVu \in \mathcal{F}_{\star,\mathbb{C}}^{s+2,\infty} \cap \mathcal{Q}_n. \quad (16)$$

In particular, one has  $\check{v}_{u,\lambda} = Vv_{u,\lambda}$ .  $\times$

*Remark 2.6.* By the same approach, one can study the inhomogeneous equation

$$(L - \lambda)f = g, \quad g \in \mathcal{F}_{\star,\mathbb{C}}^{s,\infty},$$

for  $\lambda \in S_n$  and  $n \geq n_s$ . Writing  $f = u + v$  and  $g = P_n g + Q_n g$ , the  $Q$ -equation becomes

$$A_\lambda v = Q_n V(u + v) - Q_n g = Q_n Vv + Q_n(Vu - g)$$

leading for any given  $u \in \mathcal{P}_n$  and  $\lambda \in S_n$  to the unique solution  $\check{v}$  of the equation corresponding to (11)

$$\check{v} = Vv = K_n T_n(Vu - g) \in \mathcal{F}_{\star,\mathbb{C}}^{s,\infty},$$

and, in turn, to the unique solution  $v \in \mathcal{F}_{\star,\mathbb{C}}^{s+2,\infty} \cap \mathcal{Q}_n$  of the  $Q$ -equation

$$v = A_\lambda^{-1}Q_n(Vu - g) + A_\lambda^{-1}Q_nK_nT_n(Vu - g) = A_\lambda^{-1}Q_nK_n(Vu - g). \quad \rightarrow$$

## 2.2 Reduction

In a next step we study the  $P$ -equation  $A_\lambda u = P_n V(u+v)$  of (8). For  $n \geq n_s(q)$ ,  $u \in \mathcal{P}_n$ , and  $\lambda \in S_n$ , substitute in it the solution  $\check{v}_{u,\lambda}$  of (11), given by (15),

$$A_\lambda u = P_n V u + P_n \check{v}_{u,\lambda} = P_n (\text{Id} + K_n T_n) V u.$$

Using that  $\text{Id} + K_n T_n = K_n$  one then obtains  $A_\lambda u = P_n K_n V u$  or  $B_n u = 0$ , where

$$B_n \equiv B_n(\lambda): \mathcal{P}_n \rightarrow \mathcal{P}_n, \quad u \mapsto (A_\lambda - P_n K_n(\lambda) V) u. \quad (17)$$

**Lemma 2.7** *Assume that  $q \in \mathcal{F}\ell_{0,\mathbb{C}}^{s,\infty}$  with  $-1/2 < s \leq 0$ . Then for any  $n \geq n_s$  with  $n_s$  given by Corollary 2.5,  $\lambda \in S_n$  is an eigenvalue of  $L(q)$  if and only if  $\det(B_n(\lambda)) = 0$ .  $\times$*

*Proof.* Assume that  $\lambda \in S_n$  is an eigenvalue of  $L = L(q)$ . By Lemma C.2 there exists  $0 \neq f \in \mathcal{F}\ell_{*,\mathbb{C}}^{s+2,\infty}$  so that  $Lf = \lambda f$ . Decomposing  $f = u + v \in \mathcal{P}_n \oplus \mathcal{Q}_n$  it follows by the considerations above and the assumption  $n \geq n_s$  that  $u \neq 0$  and  $B_n(\lambda)u = 0$ . Conversely, assume that  $\det(B_n(\lambda)) = 0$  for some  $\lambda \in S_n$ . Then there exists  $0 \neq u \in \mathcal{P}_n$  so that  $B_n(\lambda)u = 0$ . Since  $n \geq n_s$ , there exist  $\check{v}_{u,\lambda}$  and  $v_{u,\lambda}$  as in (15) and (16), respectively. Then  $v \equiv v_{u,\lambda} \in \mathcal{F}\ell_{*,\mathbb{C}}^{s+2,\infty} \cap \mathcal{Q}_n$  solves the  $Q$ -equation by Corollary 2.5. To see that  $u$  solves the  $P$ -equation, note that  $B_n(\lambda)u = 0$  implies that

$$A_\lambda u = P_n K_n V u = P_n (\text{Id} + K_n T_n) V u.$$

As by (15),  $\check{v}_{u,\lambda} = K_n T_n V u$ , and as  $\check{v}_{u,\lambda} = V v$ , one sees that indeed

$$A_\lambda u = P_n V u + P_n V v. \quad \blacksquare$$

*Remark 2.8.* Solutions of the inhomogeneous equation  $(L - \lambda)f = g$  for  $g \in \mathcal{F}\ell_{*,\mathbb{C}}^{s,\infty}$ ,  $\lambda \in S_n$ , and  $n \geq n_s$  can be obtained by substituting into the  $P$ -equation

$$A_\lambda u = P_n V u + P_n V v - P_n g$$

the expression for  $V v$  obtained in Remark 2.6,  $V v = K_n T_n (V u - g)$ , to get

$$\begin{aligned} A_\lambda u &= P_n V u + P_n K_n T_n V u - P_n g - P_n K_n T_n g \\ &= P_n (\text{Id} + K_n T_n) V u - P_n (\text{Id} + K_n T_n) g. \end{aligned}$$

Using that  $\text{Id} + K_n T_n = K_n$  one concludes that

$$B_n(\lambda)u = -P_n K_n(\lambda)g. \quad (18)$$

Conversely, for any solution  $u$  of (18),  $f = u + v$ , with  $v$  being the element in  $\mathcal{F}\ell_{*,\mathbb{C}}^{s+2,\infty}$  given in Remark 2.6, satisfies  $(L - \lambda)f = g$  and  $f \in \mathcal{F}\ell_{*,\mathbb{C}}^{s+2,\infty}$ .  $\rightarrow$

We denote the matrix representation of a linear operator  $F: \mathcal{P}_n \rightarrow \mathcal{P}_n$  with respect to the orthonormal basis  $e_n, e_{-n}$  of  $\mathcal{P}_n$  also by  $F$ ,

$$F = \begin{pmatrix} \langle Fe_n, e_n \rangle & \langle Fe_{-n}, e_n \rangle \\ \langle Fe_n, e_{-n} \rangle & \langle Fe_{-n}, e_{-n} \rangle \end{pmatrix}.$$

In particular,

$$A_\lambda = \begin{pmatrix} \lambda - n^2\pi^2 & 0 \\ 0 & \lambda - n^2\pi^2 \end{pmatrix}, \quad P_n K_n V = \begin{pmatrix} a_n & b_n \\ b_{-n} & a_{-n} \end{pmatrix},$$

where for any  $\lambda \in S_n$  and  $n \geq n_s$  the coefficients of  $P_n K_n V$  are given by

$$\begin{aligned} a_n &\equiv a_n(\lambda) := \langle K_n V e_n, e_n \rangle, & a_{-n} &\equiv a_{-n}(\lambda) := \langle K_n V e_{-n}, e_{-n} \rangle, \\ b_n &\equiv b_n(\lambda) := \langle K_n V e_{-n}, e_n \rangle, & b_{-n} &\equiv b_{-n}(\lambda) := \langle K_n V e_n, e_{-n} \rangle. \end{aligned}$$

Note that for any  $\lambda \in S_n$ , the functions  $a_{\pm n}(\lambda)$  and  $b_{\pm n}(\lambda)$  have the following series expansion

$$a_{\pm n}(\lambda) = \sum_{l \geq 0} \langle T_n(\lambda)^l V e_{\pm n}, e_{\pm n} \rangle, \quad b_{\pm n}(\lambda) = \sum_{l \geq 0} \langle T_n(\lambda)^l V e_{\mp n}, e_{\pm n} \rangle. \quad (19)$$

Furthermore, by a straightforward verification it follows from the expression of  $a_n$  in terms of the representation of  $K_n = \sum_{k \geq 0} T_n(\lambda)^k$  and  $V$  in Fourier space that for any  $n \geq n_s$

$$a_n = \langle K_n V e_{-n}, e_{-n} \rangle = a_{-n}. \quad (20)$$

Hence,

$$B_n(\lambda) = \begin{pmatrix} \lambda - n^2\pi^2 - a_n(\lambda) & -b_n(\lambda) \\ -b_{-n}(\lambda) & \lambda - n^2\pi^2 - a_n(\lambda) \end{pmatrix}. \quad (21)$$

In addition, if  $q$  is real valued, then

$$a_n(\bar{\lambda}) = a_n(\lambda), \quad b_{-n}(\bar{\lambda}) = \overline{b_n(\lambda)}, \quad \lambda \in S_n. \quad (22)$$

**Lemma 2.9** Suppose  $q \in \mathcal{F}\ell_{0,\mathbb{C}}^{w,s,\infty}$  with  $-1/2 < s \leq 0$  and  $w \in \mathcal{M}$ . Then for any  $n \geq n_s$ , with  $n_s$  as in Corollary 2.5, the coefficients  $a_n(\lambda)$  and  $b_{\pm n}(\lambda)$  are analytic functions on the strip  $S_n$  and for any  $\lambda \in S_n$

$$\begin{aligned} (i) \quad & |a_n(\lambda)| \leq 2 \|T_n(\lambda)\|_{w,s,\infty;\pm n} \|q\|_{s,\infty}, \\ (ii) \quad & w_{2n} \langle 2n \rangle^s |b_{\pm n}(\lambda) - q_{\pm 2n}| \leq 2 \|T_n(\lambda)\|_{w,s,\infty;\pm n} \|q\|_{w,s,\infty}. \quad \times \end{aligned}$$

*Proof.* Let us first prove the claimed estimate for  $|b_n(\lambda) - q_{2n}|$ . Since  $\|T_n(\lambda)\|_{w,s,\infty;n} \leq 1/2$  for  $n \geq n_s$  and  $\lambda \in S_n$ , the series expansion (19) of  $b_n$  converges uniformly on  $S_n$  to an analytic function in  $\lambda$ . Moreover, we obtain from the identity  $K_n = \text{Id} + T_n K_n$

$$b_n = \langle V e_{-n}, e_n \rangle + \langle T_n K_n V e_{-n}, e_n \rangle = q_{2n} + \langle T_n K_n V e_{-n}, e_n \rangle.$$

Furthermore, for any  $f \in \mathcal{F}_{\star, \mathbb{C}}^{w, s, \infty}$  we compute

$$w_{2n} \langle 2n \rangle^s |\langle f, e_n \rangle| = w_{2n} \langle 2n \rangle^s |\langle f e_n, e_{2n} \rangle| \leq \|f e_n\|_{w, s, p} = \|f\|_{w, s, p; n}.$$

Consequently, using that  $\|T_n\|_{w, s, p; n} \leq 1/2$  and hence  $\|K_n\|_{w, s, p; n} \leq 2$ , one gets

$$\begin{aligned} w_{2n} \langle 2n \rangle^s |b_n - q_{2n}| &\leq \|T_n K_n V e_{-n}\|_{w, s, p; n} \leq 2 \|T_n\|_{w, s, p; n} \|V e_{-n}\|_{w, s, p; n} \\ &= 2 \|T_n\|_{w, s, p; n} \|q\|_{w, s, p}. \end{aligned}$$

The estimates for  $|b_{-n} - q_{-2n}|$  and  $|a_n|$  are obtained in a similar fashion.  $\blacksquare$

The following refined estimate will be needed in the proof of Lemma 2.13 in Subsection 2.4.

**Lemma 2.10** *Let  $q \in \mathcal{F}_{0, \mathbb{C}}^{w, s, \infty}$  with  $w \in \mathcal{M}$  and  $-1/2 < s \leq 0$ . Then for any  $f \in \mathcal{F}_{\star, \mathbb{C}}^{s, \infty}$  and  $\lambda \in S_n$  with  $n \geq n_s$ ,*

$$w_{2n} \langle 2n \rangle^s |\langle T_n f, e_{\pm n} \rangle| \leq c'_s \varepsilon_s(n) \|q\|_{w, s, p} \|f\|_{w, s, p; \pm n}$$

where  $c'_s \geq c_s \geq 1$  is independent of  $q$ ,  $n$ , and  $\lambda$ , and

$$\varepsilon_s(n) = \begin{cases} \frac{\log \langle n \rangle}{n}, & s = 0, \\ \frac{1}{n^{1-|s|}}, & -1/2 < s < 0. \end{cases} \quad \times$$

*Proof.* As the estimates of  $\langle T_n f, e_n \rangle$  and  $\langle T_n f, e_{-n} \rangle$  can be proved in a similar way we concentrate on  $\langle T_n f, e_n \rangle$ . Since by definition  $T_n = V A_\lambda^{-1} Q_n$ ,

$$\langle T_n f, e_n \rangle = \sum_{|m| \neq n} \frac{q_{n-m} f_m}{\lambda - m^2 \pi^2}.$$

Using that  $\langle n+m \rangle / |n+m|, \langle n-m \rangle / |n-m| \leq 2$  for  $|m| \neq n$  together with (10), and the submultiplicativity of the weight, one gets for any  $\lambda \in S_n$ ,

$$\begin{aligned} w_{2n} \langle 2n \rangle^s |\langle T_n f, e_n \rangle| &\leq 2 \sum_{|m| \neq n} \frac{\langle 2n \rangle^s}{|n^2 - m^2|^{1-|s|}} \frac{w_{n-m} |q_{n-m}|}{\langle n-m \rangle^{|s|}} \frac{w_{n+m} |f_m|}{\langle n+m \rangle^{|s|}} \\ &\leq 2 \left( \sum_{|m| \neq n} \frac{\langle 2n \rangle^s}{|n^2 - m^2|^{1-|s|}} \right) \|q\|_{w, s, \infty} \|f\|_{w, s, \infty; n}. \end{aligned}$$

Finally, by Lemma A.1,

$$\sum_{|m| \neq n} \frac{1}{|n^2 - m^2|^{1-|s|}} \leq \begin{cases} \frac{\tilde{c}_s \log \langle n \rangle}{n}, & s = 0, \\ \frac{\tilde{c}_s}{n^{1-2|s|}}, & -1/2 < s < 0. \end{cases}$$

Altogether we thus have proved the claim.  $\blacksquare$

The preceding lemma together with (14) implies that for any  $n \geq n_s$ , the function  $\det B_n(\lambda) = (\lambda - n^2\pi^2 - a_n)^2 - b_n b_{-n}$  is analytic in  $\lambda \in S_n$  and can be considered a small perturbation of  $(\lambda - n^2\pi^2)^2$  provided  $n \geq n_s$  is sufficiently large.

**Lemma 2.11** *Suppose  $q \in \mathcal{F}\ell_{0,\mathbb{C}}^{s,\infty}$  with  $-1/2 < s \leq 0$ . Choose  $n_s = n_s(q) \geq 1$  as in Corollary 2.5. Then for any  $n \geq n_s$ ,  $\det(B_n(\lambda))$  has exactly two roots  $\xi_{n,1}$  and  $\xi_{n,2}$  in  $S_n$  counted with multiplicity. They are contained in*

$$D_n := \{\lambda : |\lambda - n^2\pi^2| \leq 4n^{1/2}\} \subset S_n$$

and satisfy

$$|\xi_{n,1} - \xi_{n,2}| \leq \sqrt{6} \sup_{\lambda \in S_n} |b_n(\lambda)b_{-n}(\lambda)|^{1/2}. \quad \times \quad (23)$$

*Proof.* Since for any  $n \geq n_s$  and  $\lambda \in S_n$ ,  $\|T_n(\lambda)\|_{s,\infty;\pm n} \leq 1/2$ , one concludes from the preceding lemma that  $|a_n(\lambda)| \leq \|q\|_{s,\infty}$  and, with  $|b_{\pm n}(\lambda)| \leq |q_{\pm 2n}| + |b_{\pm n}(\lambda) - q_{\pm 2n}|$ , that

$$\langle 2n \rangle^s |b_{\pm n}(\lambda)| \leq 2\|q\|_{s,\infty}.$$

Furthermore, by (14),

$$2\|q\|_{s,\infty} \leq n_s^{1/2-|s|}.$$

Therefore, for any  $\lambda, \mu \in S_n$ ,

$$\begin{aligned} |a_n(\mu)| + |b_n(\lambda)b_{-n}(\lambda)|^{1/2} &\leq \left(1 + 2\langle 2n \rangle^{|s|}\right) \|q\|_{s,\infty} \\ &< 6n^{|s|} \|q\|_{s,\infty} \leq 4n^{1/2} = \inf_{\lambda \in \partial D_n} |\lambda - n^2\pi^2|. \end{aligned} \quad (24)$$

It then follows that  $\det B_n(\lambda)$  has no root in  $S_n \setminus D_n$ . Indeed, assume that  $\xi \in S_n$  is a root, then  $|\xi - n^2\pi^2 - a_n(\xi)| = |b_n(\xi)b_{-n}(\xi)|^{1/2}$  and hence

$$|\xi - n^2\pi^2| \leq |a_n(\xi)| + |b_n(\xi)b_{-n}(\xi)|^{1/2} < 4n^{1/2},$$

implying that  $\xi \in D_n$ . In addition, (24) implies that by Rouché's theorem the two analytic functions  $\lambda - n^2\pi^2$  and  $\lambda - n^2\pi^2 - a_n(\lambda)$ , defined on the strip  $S_n$  have the same number of roots in  $D_n$  when counted with multiplicities. As a consequence  $(\lambda - n^2\pi^2 - a_n(\lambda))^2$  has a double root in  $D_n$ . Finally, (24) also implies that

$$\begin{aligned} \sup_{\lambda \in S_n} |b_n(\lambda)b_{-n}(\lambda)|^{1/2} &< \inf_{\lambda \in \partial D_n} |\lambda - n^2\pi^2| - \sup_{\lambda \in S_n} |a_n(\lambda)| \\ &\leq \inf_{\lambda \in \partial D_n} |\lambda - n^2\pi^2 - a_n(\lambda)| \end{aligned}$$

and hence again by Rouché's theorem, the analytic functions  $(\lambda - n^2\pi^2 - a_n(\lambda))^2$  and  $(\lambda - n^2\pi^2 - a_n(\lambda))^2 - b_n(\lambda)b_{-n}(\lambda)$  have the same number of roots in  $D_n$ .

Altogether we thus have established that  $\det(B_n(\lambda)) = (\lambda - n^2\pi^2 - a_n(\lambda))^2 - b_n(\lambda)b_{-n}(\lambda)$  has precisely two roots  $\xi_{n,1}, \xi_{n,2}$  in  $D_n$ .

To estimate the distance of the roots, write  $\det B_n(\lambda)$  as a product  $g_+(\lambda)g_-(\lambda)$  where  $g_{\pm}(\lambda) = \lambda - n^2\pi^2 - a_n(\lambda) \mp \varphi_n(\lambda)$  and  $\varphi_n(\lambda) = \sqrt{b_n(\lambda)b_{-n}(\lambda)}$  with an arbitrary choice of the sign of the root for any  $\lambda$ . Each root  $\xi$  of  $\det(B_n)$  is either a root of  $g_+$  or  $g_-$  and thus satisfies

$$\xi \in \{n^2\pi^2 + a_n(\xi) \pm \varphi_n(\xi)\}.$$

As a consequence,

$$\begin{aligned} |\xi_{n,1} - \xi_{n,2}| &\leq |a_n(\xi_{n,1}) - a_n(\xi_{n,2})| + \max_{\pm} |\varphi_n(\xi_{n,1}) \pm \varphi_n(\xi_{n,2})| \\ &\leq \sup_{\lambda \in D_n} |\partial_{\lambda} a_n(\lambda)| |\xi_{n,1} - \xi_{n,2}| + 2 \sup_{\lambda \in D_n} |\varphi_n(\lambda)|. \end{aligned} \quad (25)$$

Since

$$\text{dist}(D_n, \partial S_n) \geq 12n - 4n^{1/2} \geq 8n,$$

one concludes from Cauchy's estimate and the estimate  $2\|q\|_{s,\infty} \leq n^{1/2-|s|}$  following from (14) that

$$\sup_{\lambda \in D_n} |\partial_{\lambda} a_n(\lambda)| \leq \frac{\sup_{\lambda \in S_n} |a_n(\lambda)|}{\text{dist}(D_n, \partial S_n)} \leq \frac{\|q\|_{s,\infty}}{8n} \leq \frac{1}{16}.$$

Therefore, by (25),

$$|\xi_{n,1} - \xi_{n,2}|^2 \leq 6 \sup_{\lambda \in D_n} |b_n(\lambda)b_{-n}(\lambda)|$$

as claimed.  $\blacksquare$

### 2.3 Proof of Theorem 2.1 (i)

Let  $q \in \mathcal{F}\ell_{0,\mathbb{C}}^{w,s,\infty}$  with  $-1/2 < s \leq 0$  and  $w \in \mathcal{M}$ . The eigenvalues of  $L(q)$ , when listed with lexicographic ordering, satisfy

$$\lambda_0^+ \preccurlyeq \lambda_1^- \preccurlyeq \lambda_1^+ \preccurlyeq \cdots, \quad \text{and} \quad \lambda_n^{\pm} = n^2\pi^2 + n\ell_n^2.$$

It follows from a standard counting argument that for  $n \geq n_s$  with  $n_s$  as in Corollary 2.5 that  $\lambda_n^{\pm} \in S_n$  and  $\lambda_n^{\pm} \notin S_k$  for any  $k \neq n$ . It then follows from Lemma 2.7 and Lemma 2.11 that  $\{\xi_{n,1}, \xi_{n,2}\} = \{\lambda_n^-, \lambda_n^+\}$  and hence  $\gamma_n = \lambda_n^+ - \lambda_n^-$  satisfies

$$|\gamma_n| = |\xi_{n,1} - \xi_{n,2}|, \quad \forall n \geq n_s.$$

**Lemma 2.12** *If  $q \in \mathcal{F}\ell_{0,\mathbb{C}}^{w,s,\infty}$  with  $w \in \mathcal{M}$  and  $-1/2 < s \leq 0$ , then for any  $N \geq n_s$ ,*

$$\|T_N \gamma(q)\|_{w,s,\infty} \leq 4\|T_N q\|_{w,s,\infty} + \frac{16c_s}{N^{1/2-|s|}} \|q\|_{w,s,\infty}^2. \quad \times$$

*Proof of Theorem 2.1 (i).* By Lemma 2.11,

$$\begin{aligned} |\gamma_n| &= |\xi_{n,1} - \xi_{n,2}| \leq \sqrt{3} \left( \sup_{\lambda \in S_n} |b_n(\lambda)| + \sup_{\lambda \in S_n} |b_{-n}(\lambda)| \right) \\ &\leq \sqrt{3} \left( |q_{2n}| + |q_{-2n}| + \sup_{\lambda \in S_n} |b_n(\lambda) - q_{2n}| + \sup_{\lambda \in S_n} |b_{-n}(\lambda) - q_{-2n}| \right). \end{aligned}$$

It then follows from Lemma 2.4 and Lemma 2.9 that for  $n \geq N$  with  $N := n_s$

$$w_{2n} \langle 2n \rangle^s |\gamma_n| \leq \sqrt{3} \left( w_{2n} \langle 2n \rangle^s |q_{2n}| + w_{2n} \langle 2n \rangle^s |q_{-2n}| + \frac{4c_s}{n^{1/2-|s|}} \|q\|_{w,s,\infty}^2 \right).$$

Thus,  $(\gamma_n(q))_{n \geq 1} \in \ell_{\mathbb{C}}^{w,s,\infty}(\mathbb{N})$ . As  $n_s$  can be chosen locally uniformly in  $q \in \mathcal{F}_{0,\mathbb{C}}^{w,s,\infty}$ , the map  $\mathcal{F}_{0,\mathbb{C}}^{w,s,\infty} \rightarrow \ell_{\mathbb{C}}^{w,s,\infty}(\mathbb{N})$ ,  $q \mapsto (\gamma_n(q))_{n \geq 1}$  is locally bounded. ■

## 2.4 Jordan blocks of $L(q)$

To treat the Dirichlet problem, we develop the methods of [7], where the case  $q \in \mathcal{F}_{0,\mathbb{C}}^{s,p}$  with  $-1/2 \leq s \leq 0$  and  $2 \leq p < \infty$  was considered, to the case with  $-1/2 < s \leq 0$  and  $p = \infty$ . If  $q \in \mathcal{F}_{0,\mathbb{C}}^{s,\infty}$  is not real valued, then the operator  $L(q)$  might have complex eigenvalues and the geometric multiplicity of an eigenvalue could be less than its algebraic multiplicity.

We choose  $\check{n}_s \geq n_s$ , where  $n_s$  as in Corollary 2.5, so that in addition

$$\begin{aligned} |\lambda_n^\pm| &\leq (\check{n}_s - 1)^2 \pi^2 + \check{n}_s/2, & \forall n < \check{n}_s, \\ |\lambda_n^\pm - n^2 \pi^2| &\leq n/2, & \forall n \geq \check{n}_s, \\ \lambda_n^\pm &\text{ are 1-periodic [1-antiperiodic] if } n \text{ even [odd]} & \forall n \geq \check{n}_s. \end{aligned} \quad (26)$$

Note that  $\check{n}_s$  can be chosen uniformly on bounded subsets of  $\mathcal{F}_{0,\mathbb{C}}^{w,s,\infty}$  since  $\mathcal{F}_{0,\mathbb{C}}^{w,s,\infty}$  embeds compactly into  $H_{0,\mathbb{C}}^{-1}$ . For  $n \geq \check{n}_s$  we further let

$$E_n = \begin{cases} \text{Null}(L - \lambda_n^+) \oplus \text{Null}(L - \lambda_n^-), & \lambda_n^+ \neq \lambda_n^-, \\ \text{Null}(L - \lambda_n^+)^2, & \lambda_n^+ = \lambda_n^-. \end{cases}$$

We need to estimate the coefficients of  $L(q)|_{E_n}$  when represented with respect to an appropriate orthonormal basis of  $E_n$ . In the case where  $\lambda_n^+ = \lambda_n^-$  the matrix representation will be in Jordan normal form. By Lemma C.3,  $E_n \subset \mathcal{F}_{\star,\mathbb{C}}^{s+2,\infty} \hookrightarrow L^2 := L^2([0,2], \mathbb{C})$ . Denote by  $f_n^+ \in E_n$  an  $L^2$ -normalized eigenfunction corresponding to  $\lambda_n^+$  and by  $\varphi_n$  an  $L^2$ -normalized element in  $E_n$  so that  $\{f_n^+, \varphi_n\}$  forms an  $L^2$ -orthonormal basis of  $E_n$ . Then the following lemma holds.

**Lemma 2.13** *Let  $q \in \mathcal{F}_{0,\mathbb{C}}^{s,\infty}$  with  $-1/2 < s \leq 0$ . Then there exists  $n'_s \geq \check{n}_s$  - with  $\check{n}_s$  given by (26) - so that for any  $n \geq n'_s$ ,*

$$(L - \lambda_n^+) \varphi_n = -\gamma_n \varphi_n + \eta_n f_n^+,$$



where  $\eta_n \in \mathbb{C}$  satisfies the estimate

$$|\eta_n| \leq 16(|\gamma_n| + |b_n(\lambda_n^+)| + |b_{-n}(\lambda_n^+)|).$$

The threshold  $n'_s$  can be chosen locally uniformly in  $q \in \mathcal{F}_{0,\mathbb{C}}^{s,\infty}$ .  $\times$

*Proof.* We begin by verifying the claimed formula for  $(L - \lambda_n^+)\varphi_n$  in the case where  $\lambda_n^+ \neq \lambda_n^-$ . Let  $f_n^-$  be an  $L^2$ -normalized eigenfunction corresponding to  $\lambda_n^-$ . As  $f_n^- \in E_n$  there exist  $a, b \in \mathbb{C}$  with  $|a|^2 + |b|^2 = 1$  and  $b \neq 0$  so that

$$f_n^- = af_n^+ + b\varphi_n \quad \text{or} \quad \varphi_n = \frac{1}{b}f_n^- - \frac{a}{b}f_n^+.$$

Hence

$$L\varphi_n = \frac{1}{b}\lambda_n^-f_n^- - \frac{a}{b}\lambda_n^+f_n^+.$$

Substituting the expression for  $f_n^-$  into the latter identity then leads to

$$(L - \lambda_n^+)\varphi_n = (\lambda_n^- - \lambda_n^+)\varphi_n + \frac{a}{b}(\lambda_n^- - \lambda_n^+)f_n^+ = -\gamma_n\varphi_n + \eta_nf_n^+$$

where  $\eta_n = -\gamma_na/b$ . In the case  $\lambda_n^+$  is a double eigenvalue of geometric multiplicity two,  $\varphi_n$  is an eigenfunction of  $L$  and one has  $\eta_n = 0$ . Finally, in the case  $\lambda_n^+$  is a double eigenvalue of geometric multiplicity one,  $(L - \lambda_n^+)\varphi_n$  is in the eigenspace  $E_n^+ \subset E_n$  as claimed.

To prove the claimed estimate for  $\eta_n$ , we view  $(L - \lambda_n^+)\varphi_n = -\gamma_n\varphi_n + \eta_nf_n^+$  as a linear equation with inhomogeneous term  $g = -\gamma_n\varphi_n + \eta_nf_n^+$ . By identity (18) one has

$$B_nP_n\varphi_n = \gamma_nP_nK_n\varphi_n - \eta_nP_nK_nf_n^+,$$

where  $K_n \equiv K_n(\lambda_n^+)$  and  $B_n \equiv B_n(\lambda_n^+)$ . To estimate  $\eta_n$ , take the  $L^2$ -inner product of the latter identity with  $P_nf_n^+$  to get

$$\eta_n\langle P_nK_nf_n^+, P_nf_n^+ \rangle = \gamma_nI - II, \tag{27}$$

where

$$I = \langle P_nK_n\varphi_n, P_nf_n^+ \rangle, \quad II = \langle B_nP_n\varphi_n, P_nf_n^+ \rangle.$$

We begin by estimating  $\langle P_nK_nf_n^+, P_nf_n^+ \rangle$ . Using that  $K_n = \text{Id} + T_nK_n$  one gets

$$\langle P_nK_nf_n^+, P_nf_n^+ \rangle = \|P_nf_n^+\|_{L^2}^2 + \langle T_nK_nf_n^+, P_nf_n^+ \rangle,$$

and by Cauchy-Schwarz

$$|\langle T_nK_nf_n^+, P_nf_n^+ \rangle| \leq \left( \sum_{m \in \{\pm n\}} |\langle T_nK_nf_n^+, e_m \rangle|^2 \right)^{1/2} \|P_nf_n^+\|_{L^2}.$$

Note that  $\|P_n f_n^+\|_{L^2} \leq \|f_n^+\|_{L^2} = 1$ . Moreover, by Lemma 2.10 one has

$$|\langle T_n K_n f_n^+, e_{\pm n} \rangle| \leq \begin{cases} \frac{\log n}{n} C_s \|q\|_{s,\infty} \|K_n f_n^+\|_{s,\infty;\pm n}, & s = 0, \\ \frac{1}{n^{1-2|s|}} C_s \|q\|_{s,\infty} \|K_n f_n^+\|_{s,\infty;\pm n}, & -1/2 < s < 0. \end{cases}$$

By Corollary 2.5,  $\|K_n\|_{s,\infty;n} \leq 2$  and as  $L_{\mathbb{C}}^2[0,2] \hookrightarrow \mathcal{F}\ell_{*,\mathbb{C}}^{s,\infty}$ ,  $\|f_n^+\|_{s,\infty;\pm n} \leq \|f_n^+\|_{0,2;\pm n} = 1$ . Hence there exists  $n'_s \geq \check{n}_s$  so that

$$|\langle T_n K_n f_n^+, P_n f_n^+ \rangle| \leq \frac{1}{8}.$$

By increasing  $n'_s$  if necessary, Lemma 2.14 below assures that  $\|P_n f_n^+\|_{L^2} \geq 1/2$ . Thus the left hand side of (27) can be estimated as follows

$$|\eta_n \langle P_n K_n f_n^+, P_n f_n^+ \rangle| \geq |\eta_n| \left( \frac{1}{4} - \frac{1}{8} \right) = \frac{1}{8} |\eta_n|, \quad \forall n \geq n'_s. \quad (28)$$

Next let us estimate the term  $I = \langle P_n K_n \varphi_n, P_n f_n^+ \rangle$  in (27). Using again  $K_n = \text{Id} + T_n K_n$  one sees that

$$I = \langle P_n \varphi_n, P_n f_n^+ \rangle + \langle T_n K_n \varphi_n, P_n f_n^+ \rangle.$$

Clearly,  $|\langle P_n \varphi_n, P_n f_n^+ \rangle| \leq \|\varphi_n\|_{L^2} \|f_n^+\|_{L^2} \leq 1$  and arguing as above for the second term, one then concludes that

$$|I| \leq 1 + 1/8, \quad \forall n \geq n'_s. \quad (29)$$

Finally it remains to estimate  $II = \langle B_n P_n \varphi_n, P_n f_n^+ \rangle$ . Using again  $\|\varphi_n\|_{L^2} = \|f_n^+\|_{L^2} = 1$ , we conclude from the matrix representation (21) of  $B_n$  that

$$|\langle B_n P_n \varphi_n, P_n f_n^+ \rangle| \leq \|B_n\| \|\varphi_n\|_{L^2} \|f_n^+\|_{L^2} \leq |\lambda_n^+ - n^2 \pi^2 - a_n| + |b_n| + |b_{-n}|.$$

Since  $\det B_n(\lambda_n^+) = 0$ , one has

$$|\lambda_n^+ - n^2 \pi^2 - a_n| = |b_n b_{-n}|^{1/2} \leq \frac{1}{2} (|b_n| + |b_{-n}|),$$

and hence it follows that for all  $n \geq n'_s$  that

$$|II| \leq 2(|b_n| + |b_{-n}|). \quad (30)$$

Combining (28)-(30) leads to the claimed estimate for  $\eta_n$ .  $\blacksquare$

It remains to prove the estimate of  $P_n$  used in the proof of Lemma 2.13. To this end, we introduce for  $n \geq n_s$  the Riesz projector  $P_{n,q}: L^2 \rightarrow E_n$  given by (see also Appendix C)

$$P_{n,q} = \frac{1}{2\pi i} \int_{|\lambda - n^2 \pi^2| = n} (\lambda - L(q))^{-1} d\lambda.$$

**Lemma 2.14** *Let  $q \in \mathcal{F}\ell_{0,\mathbb{C}}^{s,\infty}$  with  $-1/2 < s \leq 0$ . Then there exists  $\tilde{n}_s \geq \check{n}_s$  – with  $\check{n}_s$  given by (26) – so that for any eigenfunction  $f \in \mathcal{F}\ell_{*,\mathbb{C}}^{s+2,p}$  of  $L(q)$  corresponding to an eigenvalue  $\lambda \in S_n$  with  $n \geq \tilde{n}_s$ ,*

$$\|P_n f\|_{L^2} \geq \frac{1}{2} \|f\|_{L^2}.$$

*The threshold  $\tilde{n}_s$  can be chosen locally uniformly for  $q$ .  $\times$*

*Proof.* In Lemma C.4 we show that as  $n \rightarrow \infty$ ,

$$\|P_{n,q} - P_n\|_{L^2 \rightarrow L^\infty} = o(1),$$

locally uniformly in  $q \in \mathcal{F}\ell_{0,\mathbb{C}}^{s,\infty}$ . Clearly,  $P_{n,q} f = f$ , hence

$$\|P_n f\|_{L^2} \geq \|P_{n,q} f\|_{L^2} - \|(P_{n,q} - P_n) f\|_{L^2} \geq (1 + o(1)) \|f\|_{L^2}. \quad \blacksquare$$

## 2.5 Proof of Theorem 2.1 (ii)

We begin with a brief outline of the proof of Theorem 2.1 (ii). Let  $q \in \mathcal{F}\ell_{0,\mathbb{C}}^{s,\infty}$  with  $-1/2 < s \leq 0$ . Since according to [16] for any  $q \in H_{0,\mathbb{C}}^{-1}$  the Dirichlet eigenvalues, when listed in lexicographical ordering and with their algebraic multiplicities,  $\mu_1 \preceq \mu_2 \preceq \dots$ , satisfy the asymptotics  $\mu_n = n^2 \pi^2 + n \ell_n^2$ , they are simple for  $n \geq n_{\text{dir}}$ , where  $n_{\text{dir}} \geq 1$  can be chosen locally uniformly for  $q \in H_{0,\mathbb{C}}^{-1}$ . For any  $n \geq n_{\text{dir}}$  let  $g_n$  be an  $L^2$ -normalized eigenfunction corresponding to  $\mu_n$ . Then

$$g_n \in H_{\text{dir},\mathbb{C}}^1 := \{g \in H^1([0,1], \mathbb{C}) : g(0) = g(1) = 0\}.$$

Now let  $q \in \mathcal{F}\ell_{0,\mathbb{C}}^{s,\infty}$  with  $-1/2 < s \leq 0$ . Increase  $n'_s$  of Lemma 2.13, if necessary, so that  $n'_s \geq n_{\text{dir}}$  and denote by  $E_n$  the two dimensional subspace introduced in Section 2.4. We will choose an  $L^2$ -normalized function  $\tilde{G}_n$  in  $E_n$  so that its restriction  $G_n$  to the interval  $\mathcal{I} = [0,1]$  is in  $H_{\text{dir},\mathbb{C}}^1$  and close to  $g_n$ . We then show that  $\mu_n - \lambda_n^+$  can be estimated in terms of  $\langle (L_{\text{dir}} - \lambda_n^+) G_n, G_n \rangle_{\mathcal{I}}$ , where  $\langle f, g \rangle_{\mathcal{I}}$  denotes the  $L^2$ -inner product on  $\mathcal{I}$ ,  $\langle f, g \rangle_{\mathcal{I}} = \int_0^1 f(x) \overline{g(x)} \, dx$ . As by Lemma 2.13

$$(L - \lambda_n^+) \tilde{G}_n = O(|\gamma_n| + |b_n(\lambda_n^+)| + |b_{-n}(\lambda_n^+)|),$$

the claimed estimates for  $\mu_n - \tau_n = \mu_n - \lambda_n^+ + \gamma_n/2$  then follow from the estimates of  $\gamma_n$  of Theorem 2.1 (i) and the ones of  $b_n - q_{2n}$ ,  $b_{-n} - q_{-2n}$  of Lemma 2.9 (ii).

The function  $\tilde{G}_n$  is defined as follows. Let  $f_n^+$ ,  $\varphi_n$  be the  $L^2$ -orthonormal basis of  $E_n$  chosen in Section 2.4. As  $E_n \subset H_{\mathbb{C}}^1(\mathbb{R}/2\mathbb{Z})$ , its elements are continuous functions by the Sobolev embedding theorem. If  $f_n^+(0) = 0$ , then  $f_n^+(1) = 0$  as  $f_n^+$  is an eigenfunction of the 1-periodic/antiperiodic eigenvalue  $\lambda_n^+$  of  $L(q)$  and we set  $\tilde{G}_n = f_n^+$ . If  $f_n^+(0) \neq 0$ , then we define  $\tilde{G}_n(x) =$

$r_n(\varphi_n(0)f_n^+(x) - f_n^+(0)\varphi_n(x))$ , where  $r_n > 0$  is chosen in such a way that  $\int_0^1 |\tilde{G}_n(x)|^2 dx = 1$ . Then  $\tilde{G}_n(0) = \tilde{G}_n(1) = 0$  and since  $\tilde{G}_n$  is an element of  $E_n$  its restriction  $G_n := \tilde{G}_n|_{\mathcal{I}}$  is in  $H_{\text{dir}, \mathbb{C}}^1$ .

Denote by  $\Pi_{n,q}$  the Riesz projection, introduced in Appendix C,

$$\Pi_{n,q} := \frac{1}{2\pi i} \int_{|\lambda - n^2\pi^2| = n} (\lambda - L_{\text{dir}}(q))^{-1} d\lambda.$$

It has  $\text{span}(g_n)$  as its range, hence there exists  $\nu_n \in \mathbb{C}$  so that

$$\Pi_{n,q}G_n = \nu_n g_n.$$

**Lemma 2.15** *Let  $q \in \mathcal{F}_{0,\mathbb{C}}^{s,\infty}$  with  $-1/2 < s \leq 0$ . Then there exists  $n_s'' \geq n_s'$  with  $n_s'$  as in Lemma 2.13 so that for any  $n \geq n_s''$*

$$\nu_n(\mu_n - \lambda_n^+)g_n = \beta_n(\eta_n \Pi_{n,q}(f_n^+|_{\mathcal{I}}) - \gamma_n \Pi_{n,q}(\varphi_n|_{\mathcal{I}})), \quad (31)$$

where  $\beta_n \in \mathbb{C}$  with  $|\beta_n| \leq 1$  and  $\eta_n$  is the off-diagonal coefficient in the matrix representation of  $(L - \lambda_n^+)|_{E_n}$  with respect to the basis  $\{f_n^+, \varphi_n\}$ , introduced in Lemma 2.13, and

$$1/2 \leq |\nu_n| \leq 3/2. \quad (32)$$

$n_s''$  can be chosen locally uniformly for  $q \in \mathcal{F}_{0,\mathbb{C}}^{s,\infty}$ .  $\times$

*Proof.* Write  $G_n = \nu_n g_n + h_n$ , where  $h_n = (\text{Id} - \Pi_{n,q})G_n$ . Then

$$(L_{\text{dir}} - \lambda_n^+)G_n = \nu_n(\mu_n - \lambda_n^+)g_n + (L_{\text{dir}} - \lambda_n^+)h_n.$$

On the other hand,  $G_n = \tilde{G}_n|_{\mathcal{I}}$ , where  $\tilde{G}_n \in E_n$  is given by  $\tilde{G}_n = \alpha_n f_n^+ + \beta_n \varphi_n$  with  $\alpha_n, \beta_n \in \mathbb{C}$  satisfying  $|\alpha_n|^2 + |\beta_n|^2 = 1$  and  $G_n \in H_{\text{dir}, \mathbb{C}}^1$ . Hence by Lemma C.1 and Lemma 2.13, for  $n \geq n_s'$ ,

$$(L_{\text{dir}} - \lambda_n^+)G_n = (L - \lambda_n^+)\tilde{G}_n|_{\mathcal{I}} = \beta_n(\eta_n f_n^+ - \gamma_n \varphi_n)|_{\mathcal{I}}.$$

Combining the two identities and using that  $\Pi_{n,q}h_n = 0$  and that  $\Pi_{n,q}$  commutes with  $(L_{\text{dir}} - \lambda_n^+)$ , one obtains, after projecting onto  $\text{span}(g_n)$ , identity (31).

It remains to prove (32). Taking the inner product of  $\Pi_{n,q}G_n = \nu_n g_n$  with  $g_n$  one gets

$$\nu_n = \nu_n \langle g_n, g_n \rangle_{\mathcal{I}} = \langle \Pi_{n,q}G_n, g_n \rangle_{\mathcal{I}}.$$

Let  $s_n(x) = \sqrt{2} \sin(n\pi x)$  and denote by  $\Pi_n = \Pi_{n,0}$  the orthogonal projection onto  $\text{span}\{s_n\}$ . Recall that  $P_{n,q}: L^2 \rightarrow E_n$  is the Riesz projection onto  $E_n$ . In Lemma C.4 we show that as  $n \rightarrow \infty$ ,

$$\|\Pi_{n,q} - \Pi_n\|_{L^2(\mathcal{I}) \rightarrow L^\infty(\mathcal{I})}, \|P_{n,q} - P_n\|_{L^2 \rightarrow L^\infty} = o(1), \quad (33)$$

locally uniformly in  $q \in \mathcal{F}\ell_{0,\mathbb{C}}^{s,\infty}$ . Thus using  $\Pi_n G_n = \Pi_n(P_n \tilde{G}_n)|_{\mathcal{I}}$  and recalling that  $\|G_n\|_{L^2(\mathcal{I})}^2 = \|g_n\|_{L^2(\mathcal{I})}^2 = 1$  we obtain

$$v_n = \langle \Pi_n G_n, g_n \rangle_{\mathcal{I}} + \langle (\Pi_{n,q} - \Pi_n) G_n, g_n \rangle_{\mathcal{I}} = \langle P_n \tilde{G}_n, \Pi_n g_n \rangle_{\mathcal{I}} + o(1).$$

Moreover, it follows from (33) that uniformly in  $0 \leq x \leq 1$

$$\Pi_n g_n(x) = e^{i\phi_n} s_n(x) + o(1), \quad n \rightarrow \infty,$$

with some real  $\phi_n$ . Similarly, again by (33), uniformly in  $0 \leq x \leq 2$

$$P_n \tilde{G}_n(x) = a_n e_n(x) + b_n e_{-n}(x) + o(1), \quad n \rightarrow \infty,$$

where, since  $\|G_n\|_{L^2(\mathcal{I})} = 1$  and  $G_n(0) = 0$ , the coefficients  $a_n$  and  $b_n$  can be chosen so that

$$|a_n|^2 + |b_n|^2 = 1, \quad a_n + b_n = 0.$$

That is  $P_n \tilde{G}_n(x) = e^{i\psi_n} s_n(x) + o(1)$  with some real  $\psi_n$  and hence

$$\langle P_n \tilde{G}_n, \Pi_n g_n \rangle_{\mathcal{I}} = e^{i\psi_n - i\phi_n} \langle s_n, s_n \rangle_{\mathcal{I}} + o(1) = e^{i\psi_n - i\phi_n} + o(1), \quad n \rightarrow \infty.$$

From this we conclude

$$|\nu_n| = 1 + o(1), \quad n \rightarrow \infty.$$

Therefore,  $1/2 \leq |\nu_n| \leq 3/2$  for all  $n \geq n_s''$  provided  $n_s'' \geq n_s'$  is sufficiently large.

Going through the arguments of the proof one verifies that  $n_s''$  can be chosen locally uniformly in  $q$ .  $\blacksquare$

Lemma 2.15 allows to complete the proof of Theorem 2.1 (ii).

*Proof of Theorem 2.1 (ii).* Take the inner product of (31) with  $g_n$  and use that  $|\nu_n| \geq 1/2$  by Lemma 2.15 to conclude that

$$\frac{1}{2} |\mu_n - \lambda_n^+| \leq |\beta_n| (|\eta_n| \langle \Pi_{n,q}(f_n^+|_{\mathcal{I}}), g_n \rangle_{\mathcal{I}} + |\gamma_n| \langle \Pi_{n,q}(\varphi_n|_{\mathcal{I}}), g_n \rangle_{\mathcal{I}}). \quad (34)$$

Recall that  $|\beta_n| \leq 1$  and note that for any  $f, g \in L_{\mathbb{C}}^2(\mathcal{I})$

$$|\langle \Pi_{n,q} f, g \rangle_{\mathcal{I}}| \leq |\langle \Pi_n f, g \rangle_{\mathcal{I}}| + |\langle (\Pi_{n,q} - \Pi_n) f, g \rangle_{\mathcal{I}}| \leq (1 + o(1)) \|f\|_{L^2(\mathcal{I})} \|g\|_{L^2(\mathcal{I})},$$

where for the latter inequality we used that by Lemma C.4 (ii),  $\|\Pi_{n,q} - \Pi_n\|_{L^2(\mathcal{I}) \rightarrow L^2(\mathcal{I})} = o(1)$  as  $n \rightarrow \infty$ . Since  $\|f_n^+\|_{L^2(\mathcal{I})} = \|\varphi_n\|_{L^2(\mathcal{I})} = 1$  and  $\|g_n\|_{L^2(\mathcal{I})} = 1$ , (34) implies that

$$|\mu_n - \lambda_n^+| \leq (2 + o(1)) (|\eta_n| + |\gamma_n|)$$

yielding with Lemma 2.13 the estimate

$$|\mu_n - \tau_n| \leq (3 + o(1)) |\gamma_n| + (32 + o(1)) (|\gamma_n| + |b_n(\lambda_n^+)| + |b_{-n}(\lambda_n^+)|).$$

By Theorem 2.1 (i) and Lemma 2.9 (ii) it then follows that  $(\tau_n - \mu_n)_{n \geq 1} \in \ell_{\mathbb{C}}^{w,s,\infty}(\mathbb{N})$ . Going through the arguments of the proof one verifies that the map  $\mathcal{F}\ell_{0,\mathbb{C}}^{w,s,\infty} \rightarrow \ell_{\mathbb{C}}^{w,s,\infty}(\mathbb{N})$ ,  $q \mapsto (\tau_n - \mu_n)_{n \geq 1}$  is locally bounded.  $\blacksquare$

## 2.6 Adapted Fourier Coefficients

The bounds of the operator norm  $\|T_n\|_{w,s,\infty;n}$  and the coefficients  $a_n$  and  $b_{\pm n}$  of  $P_n K_n(\lambda)V$ ,  $\lambda \in S_n$ , obtained in Lemma 2.4 and Lemma 2.9, respectively, are uniform in  $\lambda \in S_n$  and in  $q$  on bounded subsets of  $\mathcal{F}_{0,\mathbb{C}}^{w,s,\infty}$ . In addition, they are also uniform with respect to certain ranges of  $p$  and the weight  $w$ . To give a precise statement we introduce the balls

$$B_m^{w,s,\infty} := \{q \in \mathcal{F}_{0,\mathbb{C}}^{w,s,\infty} : \|q\|_{w,s,\infty} \leq m\}, \quad B_m^{s,\infty} := \{q \in \mathcal{F}_{0,\mathbb{C}}^{s,\infty} : \|q\|_{s,\infty} \leq m\}.$$

Then according to Lemma 2.4, given  $m > 0$  and  $-1/2 < s \leq 0$ , one can choose  $N_{m,s}$  so that

$$\frac{16c'_s m}{n^{1/2-|s|}} \leq 1/2, \quad n \geq N_{m,s}, \quad (35)$$

where  $c'_s \geq c_s \geq 1$  is chosen as in Lemma 2.10. This estimate implies that

$$\|T_n(\lambda)\|_{w,s,\infty;n} \leq 1/2, \quad \forall \lambda \in S_n, \quad w \in \mathcal{M}, \quad q \in B_{2m}^{w,s,\infty}.$$

**Lemma 2.16** *Let  $-1/2 < s \leq 0$  and  $m \geq 1$ . For  $n \geq N_{m,s}$  with  $N_{m,s}$  given as in (35), the coefficients  $a_n$  and  $b_{\pm n}$  are analytic functions on  $S_n \times B_m^{s,\infty}$ . Moreover, their restrictions to  $S_n \times B_{2m}^{w,s,\infty}$  for any  $w \in \mathcal{M}$  satisfy*

$$\begin{aligned} (i) \quad & |a_n|_{S_n \times B_{2m}^{w,s,\infty}} \leq \frac{8c_s m^2}{n^{1/2-|s|}} \leq m/4. \\ (ii) \quad & w_{2n} \langle 2n \rangle^s |b_{\pm n} - q_{\pm 2n}|_{S_n \times B_{2m}^{w,s,\infty}} \leq \frac{\log \langle n \rangle}{n^{1-|s|}} 8c'_s m^2 \leq \frac{\log \langle n \rangle}{n^{1/2}} m/4. \quad \times \end{aligned}$$

*Proof.* The claimed analyticity follows from the representations (19) of  $a_n$  and  $b_{\pm n}$  and the bounds from Lemma 2.4, Lemma 2.9, and Lemma 2.10.  $\blacksquare$

**Lemma 2.17** *Let  $-1/2 < s \leq 0$  and  $m \geq 1$ . For each  $n \geq N_{m,s}$  with  $N_{m,s}$  given as in (35), there exists a unique real analytic function*

$$\alpha_n : B_{2m}^{s,\infty} \rightarrow \mathbb{C}, \quad |\alpha_n - n^2 \pi^2|_{B_{2m}^{s,\infty}} \leq \frac{8c_s m^2}{n^{1/2-|s|}} \leq m/4,$$

*such that  $\lambda - n^2 \pi^2 - a_n(\lambda, \cdot)|_{\lambda=\alpha_n} \equiv 0$  identically on  $B_{2m}^{s,\infty}$ .*  $\times$

*Proof.* We follow the proof of [21, Lemma 5]. Let  $E$  denote the space of analytic functions  $\alpha : B_{2m}^{s,\infty} \rightarrow \mathbb{C}$  with  $|\alpha - n^2 \pi^2|_{B_{2m}^{s,\infty}} \leq \frac{8c_s m^2}{n^{1/2-|s|}}$  equipped with the usual metric induced by the topology of uniform convergence. This space is complete – cf. [6, Theorem A.4]. Fix any  $n \geq N_{m,s}$  and consider on  $E$  the fixed point problem for the operator  $\Lambda_n$ ,

$$\Lambda_n \alpha := n^2 \pi^2 + a_n(\alpha, \cdot).$$

By (35), each such function satisfies

$$|\alpha - n^2 \pi^2|_{B_{2m}^{s,\infty}} \leq 2m < 4n^{1/2},$$

and hence maps the ball  $B_{2m}^{s,\infty}$  into the disc  $D_n = \{|\lambda - n^2\pi^2| \leq 4n^{1/2}\} \subset S_n$ . Therefore, by Lemma 2.16

$$|\Lambda_n \alpha - n^2\pi^2|_{B_{2m}^{s,\infty}} \leq |a_n|_{S_n \times B_{2m}^{s,\infty}} \leq \frac{8c_s m^2}{n^{1/2-|s|}},$$

meaning that  $\Lambda_n$  maps  $E$  into  $E$ . Moreover,  $\Lambda_n$  contracts by a factor  $1/4$  by Cauchy's estimate,

$$|\partial_\lambda a_n|_{D_n \times B_{2m}^{s,\infty}} \leq \frac{|a_n|_{S_n \times B_{2m}^{s,\infty}}}{\text{dist}(D_n, \partial S_n)} \leq \frac{1}{12n - 4n^{1/2}} \frac{8c_s m^2}{n^{1/2-|s|}} \leq \frac{1}{4}.$$

Hence, we find a unique fixed point  $\alpha_n = \Lambda_n \alpha_n$  with the properties as claimed.  $\blacksquare$

To simplify notation define  $\alpha_{-n} := \alpha_n$  for  $n \geq 1$ . For any given  $m \geq 1$ , define the map  $\Omega^{(m)}$  on  $B_{2m}^{s,\infty}$  by

$$\Omega^{(m)}(q) = \sum_{0 \neq |n| < M_{m,s}} q_{2n} e_{2n} + \sum_{|n| \geq M_{m,s}} b_n(\alpha_n(q), q) e_{2n},$$

where  $M_{m,s} \geq N_{m,s}$  is chosen such that

$$\sup_{n \geq M_{m,s}} \frac{8c'_s}{n^{1/2-|s|}} \leq \frac{1}{16m}. \quad (36)$$

Thus, for  $n \geq M_{m,s}$  the Fourier coefficients of the 1-periodic function  $r = \Omega^{(m)}(q)$  are  $r_{2n} = b_n(\alpha_n(q))$ ,  $r_{-2n} = b_{-n}(\alpha_{-n}(q))$ , and

$$B_n(\alpha_n(q), q) = \begin{pmatrix} 0 & -r_{2n} \\ -r_{-2n} & 0 \end{pmatrix}.$$

These new Fourier coefficients are adapted to the lengths of the corresponding spectral gaps, whence we call  $\Omega^{(m)}$  the *adapted Fourier coefficient map* on  $B_m^{s,\infty}$ .

**Proposition 2.18** *For  $-1/2 < s \leq 0$  and  $m \geq 1$ ,  $\Omega^{(m)}$  maps  $B_m^{s,\infty}$  into  $\mathcal{F}_{0,\mathbb{C}}^{s,\infty}$ . Further, for every  $w \in \mathcal{M}$ , its restriction to  $B_m^{w,s,\infty}$  is a real analytic diffeomorphism*

$$\Omega^{(m)}|_{B_m^{w,s,\infty}} : B_m^{w,s,\infty} \rightarrow \Omega^{(m)}(B_m^{w,s,\infty}) \subset \mathcal{F}_{0,\mathbb{C}}^{w,s,\infty}$$

such that

$$\sup_{q \in B_m^{w,s,\infty}} \|d_q \Omega^{(m)} - \text{Id}\|_{w,s,\infty} \leq 1/16, \quad (37)$$

and  $B_{m/2}^{w,s,p} \subset \Omega^{(m)}(B_m^{w,s,p})$ . Moreover,

$$\frac{1}{2} \|q\|_{w,s,\infty} \leq \|\Omega^{(m)}(q)\|_{w,s,\infty} \leq 2 \|q\|_{w,s,\infty}, \quad q \in B_m^{w,s,\infty}. \quad \times \quad (38)$$

*Proof.* Since  $\alpha_n$  maps  $B_{2m}^{s,\infty}$  into  $S_n$  for  $n \geq N_{m,s}$ , each coefficient  $b_n(\alpha_n(q), q)$  is well defined for  $q \in B_{2m}^{s,\infty}$ , and by Lemma 2.16

$$w_{2n} \langle 2n \rangle^s |b_n(\alpha_n) - q_{2n}|_{B_{2m}^{s,\infty}} \leq w_{2n} \langle 2n \rangle^s |b_n - q_{2n}|_{S_n \times B_{2m}^{s,\infty}} \leq \frac{8c_s m^2}{n^{1/2-|s|}}.$$

Hence the map  $\Omega^{(m)}$  is defined on  $B_{2m}^{s,\infty}$  and

$$\begin{aligned} \sup_{q \in B_{2m}^{w,s,\infty}} \|\Omega^{(m)}(q) - q\|_{w,s,\infty} &= \sup_{q \in B_{2m}^{w,s,\infty}} \sup_{|n| \geq M_{m,s}} w_{2n} \langle 2n \rangle^s |b_n(\alpha_n) - q_{2n}|_{B_{2m}^{w,s,\infty}} \\ &\leq 8c_s m^2 \sup_{n \geq M_{m,s}} \frac{1}{n^{1/2-|s|}} \\ &\leq \frac{m^2}{16m} \leq \frac{m}{16}, \end{aligned}$$

by our choice (36) of  $M_{m,s}$ . Consequently, the inverse function theorem (Lemma A.4) applies proving that  $\Omega^{(m)}$  is a diffeomorphism onto its image which covers  $B_{m/2}^{w,s,p}$ . If  $q$  is real valued, then  $\alpha_n(q)$  is real and hence by (22)  $b_{-n}(\alpha_n(q)) = -\overline{b_n(\alpha_n(q))}$  implying that  $\Omega^{(m)}(q)$  is real valued as well. Altogether we thus have proved that  $\Omega^{(m)}$  is real analytic.

Finally, we note that by Cauchy's estimate

$$\sup_{q \in B_m^{w,s,\infty}} \|d_q \Omega^{(m)} - \text{Id}\|_{w,s,\infty} \leq \frac{\sup_{q \in B_{2m}^{w,s,\infty}} \|\Omega^{(m)}(q) - q\|_{w,s,\infty}}{m} \leq \frac{1}{16},$$

hence in view of the mean value theorem for any  $q_0, q_1 \in B_m^{w,s,p}$

$$\Omega^{(m)}(q_1) - \Omega^{(m)}(q_0) = \left(1 + \int_0^1 \left(d_{(1-t)q_0 + tq_1} \Omega^{(m)} - \text{Id}\right) dt\right)(q_1 - q_0),$$

we find that

$$\frac{1}{2} \|q_1 - q_0\|_{w,s,\infty} \leq \|\Omega^{(m)}(q_1) - \Omega^{(m)}(q_0)\|_{w,s,\infty} \leq 2 \|q_1 - q_0\|_{w,s,\infty}. \quad \blacksquare$$

## 2.7 Proof of Theorem 2.2

**Proposition 2.19** *Let  $-1/2 < s \leq 0$ ,  $m \geq 1$ , and  $w \in \mathcal{M}$ . If  $q \in B_m^{s,\infty}$  and*

$$\Omega^{(m)}(q) \in B_{m/2}^{w,s,\infty},$$

*then  $q \in B_m^{w,s,\infty} \subset \mathcal{F}\ell_{0,\mathbb{C}}^{w,s,\infty}$ .  $\times$*

*Proof.* By Proposition 2.18, the map  $\Omega^{(m)}$  is defined on  $B_m^{s,\infty}$  and a real analytic diffeomorphism onto its image; for  $w \in \mathcal{M}$ , the restriction of  $\Omega^{(m)}$  to  $B_m^{w,s,\infty} \subset B_m^{s,\infty} \cap \mathcal{F}\ell_{0,\mathbb{C}}^{w,s,\infty}$  is again a real analytic diffeomorphism onto its image and by Lemma 2.18 this image contains  $B_{m/2}^{w,s,\infty}$ . Thus, if  $\Omega^{(m)}$  maps  $q \in B_m^{s,\infty}$  to

$$r = \Omega^{(m)}(q) \in B_{m/2}^{w,s,\infty},$$

then we must have

$$q = (\Omega^{(m)})^{-1}|_{B_{m/2}^{w,s,\infty}}(r) \in B_m^{w,s,\infty} \subset \mathcal{F}\ell_{0,\mathbb{C}}^{w,s,\infty}. \quad \blacksquare$$



To proceed, we want to bound the Fourier coefficients of  $r = \Omega^{(m)}(q)$  in terms of the gap lengths of  $q$ .

**Lemma 2.20** *Let  $-1/2 < s \leq 0$ ,  $m \geq 1$ , and suppose that  $q \in B_m^{s,\infty}$ ,  $r = \Omega^{(m)}(q)$ , and  $n \geq M_{m,s}$  with  $M_{m,s}$  given as in (36). If*

$$r_{-2n} \neq 0, \quad \text{and} \quad \frac{1}{9} \leq \left| \frac{r_{2n}}{r_{-2n}} \right| \leq 9,$$

then

$$|r_{2n}r_{-2n}| \leq |\gamma_n(q)|^2 \leq 9|r_{2n}r_{-2n}|. \quad \times$$

*Proof.* We follow the proof of [21, Lemma 10]. To begin, we write  $\det B_n(\lambda) = g_+(\lambda)g_-(\lambda)$  with

$$g_{\pm}(\lambda) := \lambda - n^2\pi^2 - a_n(\lambda) \mp \varphi_n(\lambda), \quad \varphi_n(\lambda) = \sqrt{b_n(\lambda)b_{-n}(\lambda)}.$$

The assumption on  $r_{\pm 2n}$  implies that  $g_{\pm}$  are continuous, even analytic, functions of  $\lambda$ . Indeed, recall that  $r_{\pm 2n} = b_{\pm n}(\alpha_n)$ , thus

$$\varphi_n(\alpha_n) = \sqrt{b_n(\alpha_n)b_{-n}(\alpha_n)} \neq 0, \quad \rho_n := |\varphi_n(\alpha_n)| > 0,$$

so we may choose  $\varphi_n(\lambda)$  as a fixed branch of the square root locally around  $\lambda = \alpha_n$ . To obtain an estimate of the domain of analyticity, we consider the disc  $D_n^{\circ} := \{\lambda : |\lambda - \alpha_n| \leq 2\rho_n\}$ . Since by assumption  $n \geq M_{m,s}$  it follows from (36) together with  $c_s \geq 1$  that

$$\frac{m}{4} \leq \frac{n^{1/2-|s|}}{128}, \quad \forall n \geq M_{m,s}.$$

Lemma 2.16 then yields

$$|a_n|_{S_n} \leq \frac{m}{4} \leq \frac{n^{1/2-|s|}}{128}. \quad (39)$$

To estimate  $|b_n|_{S_n}$  note that

$$|b_n|_{S_n} \leq \langle 2n \rangle^{|s|} (\langle 2n \rangle^s |b_n - q_{2n}|_{S_n} + \langle 2n \rangle^s |q_{2n}|).$$

Since  $q \in B_m^{s,\infty}$ ,  $\langle 2n \rangle^s |q_{2n}| \leq m$  and hence again by Lemma 2.16 one has

$$|b_n|_{S_n} \leq 2n^{|s|}(m/2 + m) \leq 4n^{|s|}m \leq \frac{n^{1/2}}{8}. \quad (40)$$

Cauchy's estimate and definition (9) of  $S_n$  then gives

$$|\partial_{\lambda} b_{\pm n}|_{D_n^{\circ}} \leq \frac{|b_{\pm n}|_{S_n}}{\text{dist}(D_n^{\circ}, \partial S_n)} \leq \frac{n^{1/2}/8}{12n - |n^2\pi^2 - \alpha_n| - 2\rho_n} \leq \frac{1}{88}, \quad (41)$$

where we used that by Lemma 2.17,  $|\alpha_n - n^2\pi^2| \leq m/4 \leq n^{1/2-|s|}/128$  and by (40),  $\rho_n \leq n^{1/2}/8$ . Note that by (39), the same estimate holds for  $\partial_{\lambda} a_n$ ,

$$|\partial_{\lambda} a_n|_{D_n^{\circ}} \leq 1/88 \quad (42)$$

Thus by the mean value theorem, for any  $\lambda \in D_n^\circ$ ,

$$|b_{\pm n}(\lambda) - b_{\pm n}(\alpha_n)|_{D_n^\circ} \leq |\partial_\lambda b_{\pm n}|_{D_n^\circ} 2\rho_n \leq \frac{1}{44}\rho_n, \quad (43)$$

implying that  $\varphi_n(\lambda)^2$  is bounded away from zero for  $\lambda \in D_n^\circ$ . Hence  $\varphi_n(\lambda)$  is analytic for  $\lambda \in D_n^\circ$ .

By Lemma 2.11,  $\det B_n(\lambda) = g_+(\lambda)g_-(\lambda)$  has precisely two roots in  $S_n$  which both are contained in  $D_n \subset S_n$ . To estimate the location of these roots, we approximate  $g_\pm(\lambda)$  by  $h_\pm(\lambda)$  defined by

$$\begin{aligned} h_+(\lambda) &= \lambda - n^2\pi^2 - a_n(\alpha_n) - \varphi_n(\alpha_n), \\ h_-(\lambda) &= \lambda - n^2\pi^2 - a_n(\alpha_n) + \varphi_n(\alpha_n). \end{aligned}$$

Since  $\alpha_n - n^2\pi^2 - a_n(\alpha_n) = 0$ , one has

$$h_+(\lambda) = \lambda - \alpha_n - \varphi_n(\alpha_n), \quad h_-(\lambda) = \lambda - \alpha_n + \varphi_n(\alpha_n).$$

Clearly,  $h_+(\lambda)$  and  $h_-(\lambda)$  each have precisely one zero  $\lambda_+ = \alpha_n + \varphi_n(\alpha_n)$  and  $\lambda_- = \alpha_n - \varphi_n(\alpha_n)$ , respectively.

We want to compare  $h_+$  and  $g_+$  on the disc

$$D_n^+ := \{\lambda : |\lambda - (\alpha_n + \varphi_n(\alpha_n))| < \rho_n/2\} \subset D_n^\circ.$$

Since  $h_+(\alpha_n + \varphi_n(\alpha_n)) = 0$ , we have

$$|h_+|_{\partial D_n^+} = |h_+(\lambda) - h_+(\alpha_n + \varphi_n(\alpha_n))|_{\partial D_n^+} = \frac{\rho_n}{2}.$$

In the sequel we show that

$$|\partial_\lambda \varphi_n|_{D_n^+} \leq \frac{4}{88}, \quad (44)$$

yielding together with (42)

$$\begin{aligned} |h_+ - g_+|_{D_n^+} &\leq |a_n(\alpha_n) - a_n(\lambda)|_{D_n^+} + |\varphi_n(\alpha_n) - \varphi_n(\lambda)|_{D_n^+} \\ &\leq \left( |\partial_\lambda a_n|_{D_n^+} + |\partial_\lambda \varphi_n|_{D_n^+} \right) 2\rho_n < \frac{\rho_n}{2} = |h_+|_{\partial D_n^+}. \end{aligned}$$

Thus, it follows from Rouché's theorem that  $g_+$  has a single root contained in  $D_n^+$ . In a similar fashion, we find that  $g_-$  has a single root contained in  $D_n^- := \{\lambda : |\lambda - (\alpha_n - \varphi_n(\alpha_n))| < \rho_n/2\}$ . Since the roots of  $g_\pm(\lambda)$  are roots of  $\det B_n(\lambda)$ , they have to coincide with  $\lambda_n^\pm$  and hence

$$\rho_n \leq |\lambda_n^+ - \lambda_n^-| \leq 3\rho_n,$$

which is the claim.

It remains to show the estimate (44) for  $\partial_\lambda \varphi_n$  on  $D_n^+$ . Note that  $\overline{D_n^+} \subset D_n^\circ$  and write

$$|b_n(\lambda)| = |b_n(\lambda)b_{-n}(\lambda)|^{1/2} \left| \frac{b_n(\lambda)}{b_{-n}(\lambda)} \right|^{1/2}.$$

By the assumption of this lemma,  $\frac{1}{9} \leq \frac{|b_n(\alpha_n)|}{|b_{-n}(\alpha_n)|} \leq 9$  and by the estimate (43),

$$\frac{1}{3}\rho_n \leq |b_n(\alpha_n)| \leq 3\rho_n, \quad |b_n(\lambda) - b_n(\alpha_n)|_{D_n^\circ} \leq \frac{\rho_n}{44}.$$

Thus, by the triangle inequality we obtain

$$\frac{41}{132}\rho_n = \left(\frac{1}{3} - \frac{1}{44}\right)\rho_n \leq |b_n^+|_{D_n^\circ} \leq \left(3 + \frac{1}{44}\right)\rho_n = \frac{133}{44}\rho_n.$$

Treating  $b_{-n}$  in an analogous way, we arrive at

$$\left|\frac{b_n^+}{b_n^-}\right|_{D_n^\circ}, \left|\frac{b_n^-}{b_n^+}\right|_{D_n^\circ} \leq 10,$$

which in view of (41) finally yields the desired estimate (44),

$$|\partial_\lambda \varphi_n|_{D_n^\circ} \leq \frac{|\partial_\lambda b_n^+|_{D_n^\circ}}{2} \left|\frac{b_n^-}{b_n^+}\right|_{D_n^\circ}^{1/2} + \frac{|\partial_\lambda b_n^-|_{D_n^\circ}}{2} \left|\frac{b_n^+}{b_n^-}\right|_{D_n^\circ}^{1/2} \leq \frac{4}{88}. \quad \blacksquare$$

**Lemma 2.21** *If  $q_0 \in H_0^{-1}$  with gap lengths  $\gamma(q_0) \in \ell^{s,\infty}$ ,  $-1/2 < s \leq 0$ , then  $\text{Iso}(q_0)$  is a  $\|\cdot\|_{s,\infty}$ -norm bounded subset of  $\mathcal{F}\ell_0^{s,\infty}$ . In particular,  $q_0$  is in  $\mathcal{F}\ell_0^{s,\infty}$ .  $\bowtie$*

*Proof.* Suppose  $q_0$  is a real valued potential in  $H_0^{-1}$  with gap lengths  $\gamma(q_0) \in \ell^{s,\infty}$  for some  $-1/2 < s \leq 0$ . We can choose  $-1/2 < \sigma < s$  and  $2 \leq p < \infty$  so that  $(s - \sigma)p > 1$  and hence  $\ell^{s,\infty} \hookrightarrow \ell^{\sigma,p}$ . Consequently, by [7, Corollary 3] we have  $q_0 \in \mathcal{F}\ell^{\sigma,p}$ . Moreover, by [7, Corollary 4] the isospectral set  $\text{Iso}(q_0)$  is compact in  $\mathcal{F}\ell^{\sigma,p}$ , hence there exists  $R > 0$  so that  $\text{Iso}(q_0)$  is contained in the ball  $B_R^{\sigma,p}$ . To prove that  $\text{Iso}(q_0)$  is a bounded subset of  $\mathcal{F}\ell^{s,\infty}$ , we choose

$$m = 4(R + \|\gamma(q_0)\|_{s,\infty}). \quad (45)$$

Further, let  $w_n = \langle n \rangle^{-\sigma+s}$ ,  $n \in \mathbb{Z}$ . Then  $w \in \mathcal{M}$ ,  $\ell^{w,\sigma,\infty} = \ell^{s,\infty}$ , and  $\gamma(q_0) \in \ell^{w,\sigma,\infty}$ , while for any  $q \in \text{Iso}(q_0)$  we have

$$q \in B_m^{\sigma,\infty}.$$

The map  $\Omega^{(m)}$  is well defined on  $B_m^{\sigma,\infty}$  and

$$r \equiv r(q) = \Omega^{(m)}(q) \in \mathcal{F}\ell_0^{\sigma,\infty}.$$

Since  $r$  is real valued, we have  $r_{-n} = \overline{r_n}$  for all  $n \in \mathbb{Z}$ . Suppose  $|n| \geq M_{m,s}$ , then it follows from Lemma 2.20 that for any  $q \in \text{Iso}(q_0)$  with  $|r_n| \neq 0$  that

$$|r_n| = |r_n r_{-n}|^{1/2} \leq |\gamma_n(q)| = |\gamma_n(q_0)|.$$

The same estimate holds true when  $|r_n| = 0$ . In particular, it follows that  $r \in \mathcal{F}\ell_0^{w,\sigma,\infty}$ . To satisfy the smallness assumption of Proposition 2.19 for  $\|r\|_{w,\sigma,\infty}$ , we modify the weight  $w$ : let  $w^\varepsilon$  be the weight defined by  $w_n^\varepsilon = \min(w_n, e^{\varepsilon|n|})$ ,

$n \in \mathbb{Z}$ . Note that  $w_{-n}^\varepsilon = w_n^\varepsilon$ ,  $w_n^\varepsilon \geq 1$ , and  $w_{|n|} \leq w_{|n|+1}$  for any  $n \in \mathbb{Z}$ . For  $\varepsilon > 0$  sufficiently small, one verifies that  $\log w_{n+m}^\varepsilon \leq \log w_n^{\varepsilon p} + \log w_m^\varepsilon$  for any  $n, m \in \mathbb{Z}$ . Thus for  $\varepsilon > 0$  sufficiently small,  $w^\varepsilon$  is submultiplicative and therefore  $w^\varepsilon \in \mathcal{M}$  – see [21, Lemma 9] for details. Moreover,

$$\begin{aligned} \|r(q)\|_{w^\varepsilon, \sigma, \infty} &\leq \sup_{|n| < M_{m, \sigma}} e^{\varepsilon 2n} \langle 2n \rangle^\sigma |q_{2n}| + \sup_{|n| \geq M_{m, \sigma}} w_{2n} \langle 2n \rangle^\sigma |r_n| \\ &\leq e^{2\varepsilon M_{m, \sigma}} \|q\|_{\sigma, \infty} + \|\gamma(q_0)\|_{w, \sigma, \infty}. \end{aligned}$$

Choosing  $\varepsilon > 0$  sufficiently small, we conclude from (45) that

$$\|r(q)\|_{w^\varepsilon, \sigma, \infty} \leq 2\|q\|_{\sigma, \infty} + \|\gamma(q_0)\|_{s, \infty} \leq m/2.$$

Thus Proposition 2.19 applies yielding  $q \in B_m^{w^\varepsilon, \sigma, \infty}$ . By the definition of  $w^\varepsilon$ ,  $w_n \neq w_n^\varepsilon$  holds for at most finitely many  $n$ , hence

$$\|q\|_{w^\varepsilon, \sigma, \infty} \leq C_\varepsilon \|q\|_{w, \sigma, \infty},$$

where the constant  $C_\varepsilon \geq 1$  depends only on  $\varepsilon$  and  $M_{m, \sigma}$ , but is independent of  $q$ . Since  $\|q\|_{w, \sigma, \infty} = \|q\|_{s, \infty}$ , it thus follows that  $\|q\|_{s, \infty} \leq C_\varepsilon m$  for all  $q \in \text{Iso}(q_0)$ . ■

*Proof of Theorem 2.2.* Suppose  $q$  is a real valued potential in  $H_0^{-1}$  with gap lengths  $\gamma(q) \in \ell^{s, \infty}$  for some  $-1/2 < s \leq 0$ . By the preceding lemma,  $\text{Iso}(q)$  is bounded in  $\mathcal{F}\ell_0^{s, \infty}$ . Moreover, by [7],  $\text{Iso}(q)$  is compact in  $\mathcal{F}\ell_0^{\sigma, p}$  for any  $2 \leq p < \infty$  and  $-1/2 \leq \sigma \leq 0$  with  $(s - \sigma)p > 1$ . Consequently,  $\text{Iso}(q)$  is weak\* compact in  $\mathcal{F}\ell_0^{s, \infty}$  by Lemma B.1. ■

### 3 Birkhoff coordinates on $\mathcal{F}\ell_0^{s, \infty}$

The aim of this section is to prove Theorem 1.4. First let us recall the results on Birkhoff coordinates on  $H_0^{-1}$  obtained in [10].

**Theorem 3.1** ([10, 15]) *There exists a complex neighborhood  $\mathcal{W}$  of  $H_0^{-1}$  within  $H_{0, \mathbb{C}}^{-1}$  and an analytic map  $\Phi: \mathcal{W} \rightarrow h_{0, \mathbb{C}}^{-1/2}$ ,  $q \mapsto (z_n(q))_{n \in \mathbb{Z}}$  with the following properties:*

- (i)  $\Phi$  is canonical in the sense that  $\{z_n, z_{-n}\} = \int_0^1 \partial_u z_n \partial_x \partial_u z_{-n} dx = i$  for all  $n \geq 1$ , whereas all other brackets between coordinate functions vanish.
- (ii) For any  $s \geq -1$ , the restriction  $\Phi|_{H_0^s}$  is a map  $\Phi|_{H_0^s}: H_0^s \rightarrow h_0^{s+1/2}$  which is a bianalytic diffeomorphism.
- (iii) The KdV Hamiltonian  $H \circ \Phi^{-1}$ , expressed in the new variables, is defined on  $h_0^{3/2}$  and depends on the action variables alone. In fact, it is a real analytic function of the actions on the positive quadrant  $\ell_+^{3,1}(\mathbb{N})$ ,

$$\ell_+^{3,1}(\mathbb{N}) := \{(I_n)_{n \geq 1} : I_n \geq 0 \ \forall n \geq 1, \sum_{n \geq 1} n^3 I_n < \infty\}. \quad \times$$

We will also need the following result (cf. [10, § 3]).

**Theorem 3.2** ([10]) *After shrinking, if necessary, the complex neighborhood  $\mathcal{W}$  of  $H_0^{-1}$  in  $H_{0,\mathbb{C}}^{-1}$  of Theorem 3.1 the following holds:*

- (i) *Let  $Z_n = \{q \in H_0^{-1} : \gamma_n^2(q) \neq 0\}$  for  $n \geq 1$ . The quotient  $I_n/\gamma_n^2$ , defined on  $H_0^{-1} \setminus Z_n$ , extends analytically to  $\mathcal{W}$  for any  $n \geq 1$ . Moreover, for any  $\varepsilon > 0$  and any  $q \in \mathcal{W}$  there exists  $n_0 \geq 1$  and an open neighborhood  $\mathcal{W}_q$  of  $q$  in  $\mathcal{W}$  so that*

$$\left| 8n\pi \frac{I_n}{\gamma_n^2} - 1 \right| \leq \varepsilon, \quad \forall n \geq n_0 \quad \forall p \in \mathcal{W}_q. \quad (46)$$

- (ii) *The Birkhoff coordinates  $(z_n)_{n \in \mathbb{Z}}$  are analytic as maps from  $\mathcal{W}$  into  $\mathbb{C}$  and fulfill locally uniformly in  $\mathcal{W}$  and uniformly for  $n \geq 1$ , the estimate*

$$|z_{\pm n}| = O\left(\frac{|\gamma_n| + |\mu_n - \tau_n|}{\sqrt{n}}\right).$$

- (iii) *For any  $q \in \mathcal{W}$  and  $n \geq 1$  one has  $I_n(q) = 0$  if and only if  $\gamma_n(q) = 0$ . In particular,  $\Phi(0) = 0$ .  $\times$*

### 3.1 Birkhoff coordinates

In [7] based on the results of [15], the restrictions of the Birkhoff map

$$\Phi: H_0^{-1} \rightarrow h_0^{-1/2}, \quad q \mapsto (z_n(q))_{n \in \mathbb{Z}}, \quad z_0(q) = 0,$$

to the Fourier Lebesgue spaces  $\mathcal{F}\ell_0^{s,p}$ ,  $-1/2 \leq s \leq 0$ ,  $2 \leq p < \infty$ , are studied. It turns out that the arguments developed in the papers [7, 13] can be adapted to prove Theorem 1.4. As a first step we extend the results in [7] for  $\mathcal{F}\ell_0^{s,p}$ ,  $-1/2 \leq s \leq 0$ ,  $2 \leq p < \infty$ , to the case  $p = \infty$ . More precisely, we prove

**Lemma 3.3** *For any  $-1/2 < s \leq 0$*

$$\Phi_{s,\infty} \equiv \Phi \Big|_{\mathcal{F}\ell_0^{s,\infty}} : \mathcal{F}\ell_0^{s,\infty} \rightarrow \ell_0^{s+1/2,\infty}, \quad q \mapsto (z_n(q))_{n \in \mathbb{Z}},$$

*is real analytic and extends analytically to an open neighborhood  $\mathcal{W}_{s,\infty}$  of  $\mathcal{F}\ell_0^{s,\infty}$  in  $\mathcal{F}\ell_{0,\mathbb{C}}^{s,\infty}$ . Its Jacobian  $d_0\Phi_{s,\infty}$  at  $q = 0$  is the weighted Fourier transform*

$$d_0\Phi_{s,\infty} : \mathcal{F}\ell_0^{s,\infty} \rightarrow \ell_0^{s+1/2,\infty}, \quad f \mapsto \left( \frac{1}{\sqrt{2\pi \max(|n|, 1)}} \langle f, e_{2n} \rangle \right)_{n \in \mathbb{Z}}$$

*with inverse given by*

$$(d_0\Phi_{s,\infty})^{-1} : \ell_0^{s+1/2,\infty} \rightarrow \mathcal{F}\ell_0^{s,\infty}, \quad (z_n)_{n \in \mathbb{Z}} \mapsto \sum_{n \in \mathbb{Z}} \sqrt{2\pi|n|} z_n e_{2n}.$$

*In particular,  $\Phi_{s,\infty}$  is a local diffeomorphism at  $q = 0$ .  $\times$*

*Proof.* The coordinate functions  $z_n(q)$  are analytic functions on the complex neighborhood  $\mathcal{W} \subset H_{0,\mathbb{C}}^{-1}$  of  $H_0^{-1}$  of Theorem 3.2. Since for any  $-1/2 < s \leq 0$ ,  $\mathcal{F}\ell_{0,\mathbb{C}}^{s,\infty} \hookrightarrow H_{0,\mathbb{C}}^{-1}$ , it follows that their restrictions to  $\mathcal{W}_{s,\infty} = \mathcal{W} \cap \mathcal{F}\ell_{0,\mathbb{C}}^{s,\infty}$  are analytic as well. Furthermore,

$$z_{\pm n}(q) = O\left(\frac{|\gamma_n(q)| + |\mu_n(q) - \tau_n(q)|}{\sqrt{n}}\right)$$

locally uniformly on  $\mathcal{W}$  and uniformly in  $n \geq 1$ . By the asymptotics of the periodic and Dirichlet eigenvalues of Theorems 2.1,  $\Phi_{s,\infty}$  maps the complex neighborhood  $\mathcal{W}_{s,\infty} := \mathcal{W} \cap \mathcal{F}\ell_{0,\mathbb{C}}^{s,\infty}$  of  $\mathcal{F}\ell_0^{s,\infty}$  into the space  $\ell_{0,\mathbb{C}}^{s+1/2,\infty}$  and is locally bounded. Using [12, Theorem A.3], one sees that  $\Phi_{s,\infty}$  is analytic. The formulas for  $d_0\Phi_{s,\infty}$  and its inverse follow from [12, Theorem 9.7] by continuity. ■

In a second step, following arguments used in [13], we prove that  $\Phi_{s,\infty}$  is onto.

**Lemma 3.4** *For any  $-1/2 < s \leq 0$ , the map  $\Phi_{s,\infty}: \mathcal{F}\ell_0^{s,\infty} \rightarrow \ell_0^{s+1/2,\infty}$  is onto.*    ✕

*Proof.* Given any  $z \in \ell_0^{s+1/2,\infty} \subset h_0^{-1/2}$ , there exists  $q \in H_0^{-1}$  so that  $\Phi(q) = z$ . Moreover, by Theorem 3.2 (i) we have for all  $n$  sufficiently large

$$\left| \frac{8n\pi I_n}{\gamma_n^2} \right| \geq \frac{1}{2}.$$

Since  $I_n = z_n z_{-n}$  and  $z \in \ell_0^{s+1/2,\infty}$ , this implies  $\gamma(q) \in \ell^{s,\infty}(\mathbb{N})$ . Using Theorem 2.2, we conclude that  $q \in \mathcal{F}\ell_0^{s,\infty}$ . Since by definition  $\Phi_{s,\infty}$  is the restriction of the Birkhoff map  $\Phi$  to  $\mathcal{F}\ell_0^{s,\infty}$ , we conclude that

$$\Phi_{s,\infty}(q) = z.$$

This completes the proof. ■

### 3.2 Isospectral sets

Recall that for any  $z \in h_0^{-1/2}$ , the torus  $\mathcal{T}_z \subset h_0^{-1/2}$  was introduced in (2).

**Lemma 3.5** *Suppose  $q \in \mathcal{F}\ell_0^{s,\infty}$  with  $-1/2 < s \leq 0$ .*

- (i)  $\text{Iso}(q)$  is bounded in  $\mathcal{F}\ell_0^{s,\infty}$ .
- (ii)  $\Phi_{s,\infty}(\text{Iso}(q)) = \mathcal{T}_{\Phi(q)}$ .
- (iii) If  $\Phi(q) \notin c_0^s = \{z \in \ell_0^{s,\infty} : \langle n \rangle^s z_n \rightarrow 0\}$ , then  $\Phi(\text{Iso}(q))$  is not compact in  $\mathcal{F}\ell_0^{s,\infty}$ .    ✕

*Proof.* (i) follows from Theorem 2.2. According to [10] the identity  $\Phi(\text{Iso}(q)) = \mathcal{T}_{\Phi(q)}$  holds for any  $q \in H_0^{-1}$  and thus implies (ii). Suppose  $q \in \mathcal{F}\ell_0^{s,\infty}$  is such that  $z = \Phi(q) \notin c_0^s$ . Then there exists  $\varepsilon > 0$  and a subsequence  $(\nu_n)_{n \geq 1} \subset \mathbb{N}$  with  $\nu_n \rightarrow \infty$  so that

$$\langle \nu_n \rangle^s |z_{\nu_n}| \geq \varepsilon, \quad \forall n \geq 1.$$

For every  $m \in \mathbb{N}$  define  $z^{(m)} \in \mathcal{T}_z$  by setting  $z_0^{(m)} = 0$  and for any  $k \geq 1$ ,  $z_{-k} = z_k^{(m)}$  and

$$z_k^{(m)} = \begin{cases} -z_k, & k = \nu_m, \\ z_k, & \text{otherwise.} \end{cases}$$

It follows that  $\|z^{(m_1)} - z^{(m_2)}\|_{s,\infty} \geq 2\varepsilon$  for all  $m_1 \neq m_2$ , hence  $\mathcal{T}_z$  is not compact. ■

### 3.3 Weak\* topology

In this subsection we establish various properties of  $\Phi_{s,\infty}$  related to the weak\* topology.

**Lemma 3.6** *For any  $-1/2 < s \leq 0$ , the map  $\Phi_{s,\infty}: \mathcal{F}\ell_0^{s,\infty} \rightarrow \ell_0^{s+1/2,\infty}$  is  $\|\cdot\|_{s,\infty}$ -norm bounded.   ✕*

*Proof.* It suffices to consider the case of the ball  $B_m^{s,\infty} \subset \mathcal{F}\ell_0^{s,\infty}$  of radius  $m \geq 1$ . Since  $B_m^{s,\infty}$  embeds compactly into  $H_0^{-1}$ , (46) implies that one can choose  $N \geq N_{m,s}$  such that for all  $q \in B_m^{s,\infty}$ ,

$$\frac{8n\pi I_n}{\gamma_n^2} \leq 2, \quad n \geq N.$$

Since  $|z_n(q)|^2 = I_n$ , we conclude with Lemma 2.12 that

$$\begin{aligned} \|T_N \Phi(q)\|_{s+1/2,\infty} &= \sup_{|n| \geq N} \langle n \rangle^{s+1/2} |z_n| \\ &\leq \sup_{|n| \geq N} \langle n \rangle^s |\gamma_n| \\ &\leq 4\|T_N q\|_{s,\infty} + \frac{16c_s}{N^{1/2-|s|}} \|q\|_{s,\infty}^2, \quad q \in B_m^{s,\infty}. \end{aligned}$$

Moreover, each of the finitely many remaining coordinate functions  $z_n(q)$ ,  $|n| < N$ , is real analytic on  $H_0^{-1}$  and hence bounded on the compact set  $B_m^{s,\infty}$ , which proves the claim. ■

**Lemma 3.7** *For any  $-1/2 < s \leq 0$ , the map  $\Phi_{s,\infty}: \mathcal{F}\ell_0^{s,\infty} \rightarrow \ell_0^{s+1/2,\infty}$  maps weak\* convergent sequences to weak\* convergent sequences.   ✕*

*Proof.* Given  $q^{(k)} \xrightarrow{*} q$  in  $\mathcal{F}\ell_0^{s,\infty}$ , there exists  $m \geq 1$  so that  $(q^{(k)})_{k \geq 1} \subset B_m^{s,\infty}$ . Since  $q^{(k)} \rightarrow q$  in  $H_0^{-1}$  and  $\Phi: H_0^{-1} \rightarrow h_0^{-1/2}$  is continuous, it follows that  $z^{(k)} := \Phi(q^{(k)}) \rightarrow \Phi(q) =: z$  in  $h_0^{-1/2}$ . In particular,  $z_n^{(k)} \rightarrow z_n$  for all  $n \in \mathbb{Z}$ . By the previous lemma it follows that  $(z^{(k)})_{k \geq 1}$  is bounded in  $\ell_0^{s+1/2,\infty}$  and hence  $z^{(k)} \xrightarrow{*} z$  in  $\ell_0^{s+1/2,\infty}$ . ■

**Corollary 3.8** *For any  $-1/2 < s \leq 0$  and  $m \geq 1$ , the map*

$$\Phi_{s,\infty}: (B_m^{s,\infty}, \sigma(\mathcal{F}\ell_0^{s,\infty}, \mathcal{F}\ell_0^{-s,1})) \rightarrow (\ell_0^{s+1/2,\infty}, \sigma(\ell_0^{s+1/2,\infty}, \ell_0^{-(s+1/2),1}))$$

*is a homeomorphism onto its image.*    ✕

*Proof.* By Lemma B.1,  $(B_m^{s,\infty}, \sigma(\mathcal{F}\ell_0^{s,\infty}, \mathcal{F}\ell_0^{-s,1}))$  is metrizable. Hence by Lemma 3.6 and Lemma 3.7, the map  $\Phi: (B_m^{s,\infty}, \sigma(\mathcal{F}\ell_0^{s,\infty}, \mathcal{F}\ell_0^{-s,1})) \rightarrow (\ell_0^{s+1/2,\infty}, \sigma(\ell_0^{s+1/2,\infty}, \ell_0^{-(s+1/2),1}))$  is continuous. Since  $\Phi_{s,\infty}: \mathcal{F}\ell_0^{s,\infty} \rightarrow \ell_0^{s+1/2,\infty}$  is bijective and  $B_m^{s,\infty}$  is compact with respect to the weak\* topology, the claim follows. ■

### 3.4 Proof of Theorem 1.4 and asymptotics of the KdV frequencies

*Proof of Theorem 1.4.* The claim follows from Lemma 3.3, Lemma 3.4, Lemma 3.5, Lemma 3.6, and Corollary 3.8. ■

Recall that in [17] the KdV frequencies  $\omega_n = \partial_{I_n} H$  have been proved to extend real analytically to  $H_0^{-1}$  – see also [11] for more recent results.

**Lemma 3.9** *Uniformly on  $\|\cdot\|_{s,\infty}$ -norm bounded subsets of  $\mathcal{F}\ell_0^{s,\infty}$ ,  $-1/2 < s \leq 0$ ,*

$$\omega_n = (2n\pi)^3 - 6I_n + o(1). \quad \times$$

*Proof.* The claim follows immediately from [11, Theorem 3.6] and the fact that  $\mathcal{F}\ell_0^{s,\infty}$  embeds compactly into  $\mathcal{F}\ell^{-1/2,p}$  if  $(s+1/2)p > 1$ . ■

## 4 Proofs of Theorems 1.1 and 1.3

*Proof of Theorem 1.1.* According to [17], for any  $q \in \mathcal{F}\ell_0^{s,\infty} \hookrightarrow H_0^{-1}$ , the solution curve  $t \mapsto S(q)(t) \in H_0^{-1}$  exists globally in time and is contained in  $\text{Iso}(q)$ . Since the latter is  $\|\cdot\|_{s,\infty}$ -norm bounded by Lemma 2.21, the solution curve is uniformly  $\|\cdot\|_{s,\infty}$ -norm bounded in time,

$$\sup_{t \in \mathbb{R}} \|S(q)(t)\|_{s,\infty} \leq \sup_{\tilde{q} \in \text{Iso}(q)} \|\tilde{q}\| < \infty.$$

By [17], any coordinate function  $t \mapsto (S(q))_n(t)$ ,  $n \in \mathbb{Z}$ , is continuous and hence  $\mathbb{R} \mapsto (\mathcal{F}\ell_0^{s,\infty}, \tau_{w*})$ ,  $t \mapsto S(q)(t)$  is a continuous map. ■



*Proof of Theorem 1.3.* Suppose  $V \subset \mathcal{F}\ell_0^{s,\infty}$  is a  $\|\cdot\|_{s,\infty}$ -norm bounded subset. Then there exists  $m \geq 1$  so that  $V \subset B_m^{s,\infty}$  and the weak\* topology induced on  $B_m^{s,\infty}$  coincides with the norm topology induced from  $\mathcal{F}\ell_0^{\sigma,p}$  provided  $(s-\sigma)p > 1$  – see Lemma B.1. Since by [7], for any  $-1/2 \leq \sigma \leq 0$ ,  $2 \leq p < \infty$ , the map

$$\mathcal{S}: (V, \|\cdot\|_{\sigma,p}) \rightarrow C([-T, T], (V, \|\cdot\|_{\sigma,p}))$$

is continuous, it follows that

$$\mathcal{S}: (V, \tau_{w*}) \rightarrow C([-T, T], (V, \tau_{w*}))$$

is continuous as well.  $\blacksquare$

*Proof of Remark 1.2.* Since by Lemma 3.3, the Birkhoff map  $\Phi$  is a local diffeomorphism near 0, it suffices to show for generic small initial data  $q$  in  $\mathcal{F}\ell_0^{s,\infty}$  that the solution curve  $t \mapsto \mathcal{S}(q)(t)$ , expressed in Birkhoff coordinates, is not continuous. But this latter claim follows in a straightforward way from the asymptotics of the KdV frequencies of Lemma 3.9.  $\blacksquare$

## 5 Wiener Algebra

It turns out that by our methods we can also prove that the KdV equation is globally in time  $C^\omega$ -wellposed on  $\mathcal{F}\ell_0^{0,1}$ , referred to as Wiener algebra. Actually, we prove such a result for any Fourier Lebesgue space  $\mathcal{F}\ell_0^{N,1}$  with  $N \in \mathbb{Z}_{\geq 0}$ .

### 5.1 Birkhoff coordinates

In a first step we prove that  $\mathcal{F}\ell_0^{N,1}$  admits global Birkhoff coordinates. More precisely, we show

**Theorem 5.1** *For any  $N \in \mathbb{Z}_{\geq 0}$ , the restriction  $\Phi_{N,1}$  of the Birkhoff map  $\Phi$  to  $\mathcal{F}\ell_0^{N,1}$  takes values in  $\ell_0^{N+1/2,1}$  and  $\Phi_{N,1}: \mathcal{F}\ell_0^{N,1} \rightarrow \ell_0^{N+1/2,1}$  is a real analytic diffeomorphism, and therefore provides global Birkhoff coordinates on  $\mathcal{F}\ell_0^{N,1}$ .  $\times$*

Before we prove Theorem 5.1, we need to review the spectral theory of the Schrödinger operator  $-\partial_x^2 + q$  for  $q \in \mathcal{F}\ell_0^{N,1}$ .

### 5.2 Spectral Theory

The spectral theory of the operator  $L(q) = -\partial_x^2 + q$  for  $q \in \mathcal{F}\ell_0^{N,1}$  was considered in [21] where the following results were shown.

**Theorem 5.2** ([21, Theorem 1 & 4]) *Let  $N \in \mathbb{Z}_{\geq 0}$ .*

(i) For any  $q \in \mathcal{F}\ell_{0,\mathbb{C}}^{N,1}$ , the sequence of gap lengths  $(\gamma_n(q))_{n \geq 1}$ , defined in (5) is in  $\ell_{\mathbb{C}}^{N,1}(\mathbb{N})$  and the map

$$\mathcal{F}\ell_{0,\mathbb{C}}^{N,1} \rightarrow \ell_{\mathbb{C}}^{N,1}(\mathbb{N}), \quad q \mapsto (\gamma_n(q))_{n \geq 1},$$

is uniformly bounded on bounded subsets.

(ii) For any  $q \in \mathcal{F}\ell_{0,\mathbb{C}}^{N,1}$ , the sequence  $(\tau_n - \mu_n(q))_{n \geq 1}$  – cf. (5)-(6) – is in  $\ell_{\mathbb{C}}^{N,1}(\mathbb{N})$  and the map

$$\mathcal{F}\ell_{0,\mathbb{C}}^{N,1} \rightarrow \ell_{\mathbb{C}}^{N,1}(\mathbb{N}), \quad q \mapsto (\tau_n(q) - \mu_n(q))_{n \geq 1},$$

is uniformly bounded on bounded subsets.  $\times$

In addition, the following spectral characterization for a potential  $q \in L^2$  to be in  $\mathcal{F}\ell_0^{N,1}$  holds.

**Theorem 5.3** ([21, Theorem 3]) *Let  $q \in L_0^2$  and assume that  $(\gamma_n(q))_{n \geq 1} \in \ell_{\mathbb{R}}^{N,1}$  for some  $N \in \mathbb{Z}_{\geq 0}$ . Then  $q \in \mathcal{F}\ell_0^{N,1}$  and  $\text{Iso}(q) \subset \mathcal{F}\ell_0^{N,1}$ .*  $\times$

### 5.3 Proof of Theorem 5.1

Theorem 5.1 will follow from the following lemmas.

**Lemma 5.4** *For any  $N \in \mathbb{Z}_{\geq 0}$*

$$\Phi_{N,1} \equiv \Phi \Big|_{\mathcal{F}\ell_0^{N,1}} : \mathcal{F}\ell_0^{N,1} \rightarrow \ell_0^{N+1/2,1}, \quad q \mapsto (z_n(q))_{n \in \mathbb{Z}},$$

is real analytic and extends analytically to an open neighborhood  $\mathcal{W}_{N,1}$  of  $\mathcal{F}\ell_0^{N,1}$  in  $\mathcal{F}\ell_{0,\mathbb{C}}^{N,1}$ .  $\times$

*Proof.* The coordinate functions  $z_n(q) = (\Phi(q))_n$ ,  $n \in \mathbb{Z}$ , are analytic functions on the complex neighborhood  $\mathcal{W} \subset H_{0,\mathbb{C}}^{-1}$  of  $H_0^{-1}$  of Theorem 3.2. Furthermore,

$$z_{\pm n}(q) = O\left(\frac{|\gamma_n(q)| + |\mu_n(q) - \tau_n(q)|}{\sqrt{n}}\right)$$

locally uniformly on  $\mathcal{W}$  and uniformly in  $n \geq 1$ . By the asymptotics of the periodic and Dirichlet eigenvalues of Theorem 5.2,  $\Phi_{N,1}$  maps the complex neighborhood  $\mathcal{W}_{N,1} := \mathcal{W} \cap \mathcal{F}\ell_{0,\mathbb{C}}^{N,1}$  of  $\mathcal{F}\ell_0^{N,1}$  into the space  $\ell_{0,\mathbb{C}}^{N+1/2,1}$  and is locally bounded. It then follows from [6, Theorem A.4] that for any  $\xi \in \ell_{0,\mathbb{C}}^{-(N+1/2),\infty}$ , the map  $q \mapsto \langle \xi, \Phi(q) \rangle$  is analytic on  $\mathcal{W} \cap \ell_{0,\mathbb{C}}^{N,1}$  implying that  $\Phi : \mathcal{W}_{N,1} \rightarrow \ell_{0,\mathbb{C}}^{N+1/2,1}$  is weakly analytic. Hence by [6, Theorem A.3],  $\Phi_{N,1}$  is analytic.  $\blacksquare$

Next, following arguments used in [13], we prove that  $\Phi_{N,1}$  is onto.

**Lemma 5.5** *For any  $N \in \mathbb{Z}_{\geq 0}$ , the map  $\Phi_{N,1} : \mathcal{F}\ell_0^{N,1} \rightarrow \ell_0^{N+1/2,1}$  is onto.*  $\times$

*Proof.* For any  $z \in \ell_0^{N+1/2,1} \subset h_0^{1/2}$ , there exists  $q \in L_0^2$  so that  $\Phi(q) = z$ . Moreover, by Theorem 3.2 (i) we have for all  $n$  sufficiently large

$$\left| \frac{8n\pi I_n}{\gamma_n^2} \right| \geq \frac{1}{2}.$$

Since  $I_n = z_n z_{-n}$  and  $z \in \ell_0^{N+1/2,1}$ , this implies  $\gamma(q) \in \ell^{N,1}(\mathbb{N})$ . Using Theorem 5.3, we conclude that  $q \in \mathcal{F}\ell_0^{N,1}$ . Since by definition  $\Phi_{N,1}$  is the restriction of the Birkhoff map  $\Phi$  to  $\mathcal{F}\ell_0^{N,1}$ , we conclude

$$\Phi_{N,1}(q) = z.$$

This completes the proof.  $\blacksquare$

**Lemma 5.6** *For any  $q \in \mathcal{F}\ell_0^{N,1}$  with  $N \in \mathbb{Z}_{\geq 0}$ ,*

$$d_q \Phi_{N,1}: \mathcal{F}\ell_0^{N,1} \rightarrow \ell_0^{N+1/2,1}$$

*is a linear isomorphism.*  $\rtimes$

*Proof.* By Theorem 3.1,  $d_q \Phi: H_0^{-1} \rightarrow h_0^{-1/2}$  is a linear isomorphism for any  $q \in H_0^{-1}$ . Since  $d_q \Phi_{N,1} = d_q \Phi|_{\mathcal{F}\ell_0^{N,1}}$  for any  $q \in \mathcal{F}\ell_0^{N,1}$ , it follows from Lemma 5.4 that  $d_q \Phi_{N,1}: \mathcal{F}\ell_0^{N,1} \rightarrow \ell_0^{N+1/2,1}$  is one-to one. To show that  $d_q \Phi_{N,1}$  is onto, note that by Theorem 3.1,  $d_0 \Phi_{N,1}: \mathcal{F}\ell_0^{N,1} \rightarrow \ell_0^{N+1/2,1}$  is a weighted Fourier transform and hence a linear isomorphism. It therefore suffices to show that  $d_q \Phi_{N,1} - d_0 \Phi_{N,1}: \mathcal{F}\ell_0^{N,1} \rightarrow \ell_0^{N+1/2,1}$  is a compact operator implying that  $d_q \Phi_{N,1}$  is a Fredholm operator of index zero and thus a linear isomorphism. To show that  $d_q \Phi_{N,1} - d_0 \Phi_{N,1}: \mathcal{F}\ell_0^{N,1} \rightarrow \ell_0^{N+1/2,1}$  is compact we use that by [14, Theorem 1.4], for any  $q \in H_0^N$ , the restriction of  $d_q \Phi$  to  $H_0^N$  has the property that  $d_q \Phi - d_0 \Phi: H_0^N \rightarrow h_0^{N+3/2}$  is a bounded linear operator. In view of the fact that  $\mathcal{F}\ell_0^{N,1} \hookrightarrow H_0^N$  is bounded and  $h_0^{N+3/2} \hookrightarrow_c \ell_0^{N+1/2,1}$  is compact, it follows that

$$d_q \Phi_{N,1} - d_0 \Phi_{N,1}: \mathcal{F}\ell_0^{N,1} \rightarrow \ell_0^{N+1/2,1}$$

is a compact operator.  $\blacksquare$

## 5.4 Frequencies

Finally, we need to consider the KdV frequencies introduced in Subsection 3.4. They are viewed either as functions on  $\mathcal{F}\ell_0^{N,1}$  or as functions of the Birkhoff coordinates on  $\ell_0^{N+1/2,1}$ .

**Lemma 5.7** *The KdV frequencies  $\omega_n$ ,  $n \geq 1$ , admit a real analytic extension to a common complex neighborhood  $\mathcal{W}^{N,1}$  of  $\mathcal{F}\ell_0^{N,1}$ ,  $N \in \mathbb{Z}_{\geq 0}$ , and for any  $r > 1$  have the asymptotic behavior*

$$\omega_n - 8n^3\pi^3 = \ell_n^r,$$

*locally uniformly on  $\mathcal{W}^{N,1}$ .*  $\rtimes$

*Proof.* Since  $\mathcal{F}\ell_{0,\mathbb{C}}^{N,1} \hookrightarrow L_{0,\mathbb{C}}^2$  this is an immediate consequence of [11, Theorem 3.6]. ■

### 5.5 Wellposedness

We are now in position to prove that the KdV equation is globally in time  $C^\omega$ -wellposed on  $\mathcal{F}\ell_0^{N,1}$  for any  $N \in \mathbb{Z}_{\geq 0}$ . First we consider the KdV equation in Birkhoff coordinates. Let  $\mathcal{S}_\Phi: (t, z) \mapsto (\varphi_n^t(z))_{n \in \mathbb{Z}}$  denote the flow in Birkhoff coordinates with coordinate functions

$$\varphi_n^t(z) = e^{i\omega_n(z)t} z_n, \quad n \in \mathbb{Z}.$$

**Lemma 5.8** *For any  $N \in \mathbb{Z}_{\geq 0}$  and  $T > 0$ , the map*

$$\mathcal{S}_\Phi: \ell_0^{N+1/2,1} \rightarrow C([-T, T], \ell_0^{N+1/2,1}), \quad z \mapsto (t \mapsto \mathcal{S}_\Phi(t, z)),$$

*is real-analytic.*    ✕

*Proof.* Since  $\omega_n - 8n^3\pi^3 = o(1)$  locally uniformly, this is an immediate consequence of [11, Theorem E.1]. ■

**Theorem 5.9** *For any  $N \in \mathbb{Z}_{\geq 0}$ , the KdV equation is globally in time  $C^\omega$ -wellposed on  $\mathcal{F}\ell_0^{N,1}$ . More precisely, for any  $T > 0$ , the map*

$$\mathcal{S}: \mathcal{F}\ell_0^{N,1} \rightarrow C([-T, T], \mathcal{F}\ell_0^{N,1}), \quad q \mapsto (t \mapsto \mathcal{S}(t, q)),$$

*is real analytic.*    ✕

*Proof.* The claim follows immediately from the Lemma 5.8 and the fact established in Theorem 5.1 that the Birkhoff map is a real analytic diffeomorphism  $\Phi: \mathcal{F}\ell_0^{N,1} \rightarrow \ell_0^{N+1/2,1}$ . ■

## A Auxiliaries

**Lemma A.1** *For any  $1/2 < \sigma < \infty$  there exists a constant  $C_\sigma > 0$  so that for any  $n \geq 1$ ,  $\sum_{|m| \neq n} \frac{1}{|m^2 - n^2|^\sigma}$  is bounded by  $C_\sigma/n^{2\sigma-1}$  if  $1/2 < \sigma < 1$ ,  $C_\sigma \frac{\log(n)}{n}$  if  $\sigma = 1$ , and  $C_\sigma/n^\sigma$  if  $\sigma > 1$ .*    ✕

*Proof.* [7, Lemma A.1]. ■

For any  $s \in \mathbb{R}$  and  $1 \leq p \leq \infty$  denote by  $\ell_{\mathbb{C}}^{s,p} \equiv \ell^{s,p}(\mathbb{Z}, \mathbb{C})$  the sequence space

$$\ell_{\mathbb{C}}^{s,p} = \{z = (z_k)_{k \in \mathbb{Z}} \subset \mathbb{C} : \|z\|_{s,p} < \infty\}.$$

**Lemma A.2** *Suppose  $-1/2 < s \leq 0$ . For any  $-1 \leq \sigma < s$  and  $2 \leq p < \infty$  with  $(s - \sigma)p > 1$  one has  $\ell_{\mathbb{C}}^{s,\infty} \hookrightarrow \ell_{\mathbb{C}}^{\sigma,p}$  and the embedding is compact. In particular, for any  $\varepsilon > 0$ ,  $\ell_{\mathbb{C}}^{s,\infty} \hookrightarrow h_{\mathbb{C}}^{-1/2+s-\varepsilon}$ .*    ✕

*Proof.* By Hölder's inequality

$$\left( \sum_{m \in \mathbb{Z}} \langle m \rangle^{\sigma p} |a_m|^p \right)^{1/p} \leq \left( \sup_{m \in \mathbb{Z}} \langle m \rangle^s |a_m| \right) \left( \sum_{m \in \mathbb{Z}} \langle m \rangle^{-(s-\sigma)p} \right)^{1/2},$$

provided  $(s - \sigma)p > 1$ . Hence  $\ell_{\mathbb{C}}^{s,\infty} \hookrightarrow \ell_{\mathbb{C}}^{\sigma,p}$ . The compactness follows from the well known characterization of compact subsets in  $\ell^p$ . ■

The following result is well known – cf. [7, Lemma 20].

**Lemma A.3** (i) Let  $-1 \leq t < -1/2$ . For  $a = (a_m)_{m \in \mathbb{Z}} \in h_{\mathbb{C}}^t$  and  $b = (b_m)_{m \in \mathbb{Z}} \in h_{\mathbb{C}}^1$ , the convolution  $a * b = (\sum_{m \in \mathbb{Z}} a_{n-m} b_m)_{n \in \mathbb{Z}}$  is well defined and

$$\|a * b\|_{t,2} \leq C_t \|a\|_{t,2} \|b\|_{1,2}.$$

(ii) Let  $-1/2 \leq s \leq 0$  and  $-s - 3/2 < t < 0$ . For any  $a = (a_m)_{m \in \mathbb{Z}} \in \ell_{\mathbb{C}}^{s,\infty}$  and  $b = (b_m)_{m \in \mathbb{Z}} \in h_{\mathbb{C}}^{t+2}$ ,

$$\|a * b\|_{s,\infty} \leq C_{s,t} \|a\|_{s,\infty} \|b\|_{t+2,2}. \quad \times$$

The following result is a version of the inverse function theorem.

**Lemma A.4** Let  $E$  be a complex Banach space and denote for  $r > 0$ ,  $B_r = \{x \in E : \|x\| \leq r\}$ . If  $f: B_m \rightarrow E$  is analytic for some  $m \geq 1$ , and

$$\sup_{x \in B_m} |f(x) - x| \leq m/8,$$

then  $f$  is an analytic diffeomorphism onto its image, and this image covers  $B_{m/2}$ .     $\times$

## B Facts on the weak \* topology

In this subsection we collect various properties of the weak\* topology  $\tau_{w*} = \sigma(\ell_0^{s,\infty}, \ell_0^{-s,1})$  on  $\ell_0^{s,\infty}$  needed in the course of the paper.

**Lemma B.1** Let  $s \in \mathbb{R}$ .

- (i) The closed unit ball of  $\ell_0^{s,\infty}$  is weak\* compact, weak\* sequentially compact, and the topology induced by the weak\* topology on this ball is metrizable.
- (ii) For any sequence  $(x^{(m)})_{m \geq 1} \subset \ell_0^{s,\infty}$  and  $x \in \ell_0^{s,\infty}$  the following statements are equivalent:

- (a)  $(x^{(m)})$  is weak\* convergent to  $x$ .
- (b)  $(x^{(m)})$  is  $\|\cdot\|_{s,\infty}$ -norm bounded and componentwise convergent, i.e.

$$\sup_{m \geq 1} \|x^{(m)}\|_{s,\infty} < \infty, \quad \lim_{m \rightarrow \infty} x_n^{(m)} = x_n, \quad \forall n \in \mathbb{Z}.$$

(c)  $(x^{(m)})$  is  $\|\cdot\|_{s,\infty}$ -norm bounded and  $x^{(m)} \rightarrow x$  in  $\ell_0^{\sigma,p}$  for some  $\sigma < s$  and  $1 \leq p < \infty$  with  $(s - \sigma)p > 1$ .

(iii) For any subset  $A \subset \ell_0^{s,\infty}$  the following statements are equivalent:

(a)  $A$  is weak\* compact.

(b)  $A$  is weak\* sequentially compact.

(c)  $A$  is  $\|\cdot\|_{s,\infty}$ -norm bounded and weak\* closed.

(d)  $A$  is  $\|\cdot\|_{s,\infty}$ -norm bounded and  $A$  is a compact subset of  $\ell_0^{\sigma,p}$  for some  $\sigma < s$  and  $1 \leq p < \infty$  with  $(s - \sigma)p > 1$ .

(iv) On any  $\|\cdot\|_{s,\infty}$ -norm bounded subset  $A \subset \ell_0^{s,\infty}$ , the topology induced by the  $\|\cdot\|_{s,p}$ -norm, provided  $(s - \sigma)p > 1$ , coincides with the topology induced by the weak\* topology of  $\ell_0^{s,\infty}$ .  $\times$

## C Schrödinger Operators

In this appendix we review definitions and properties of Schrödinger operators  $-\partial_x^2 + q$  with a singular potential  $q$  used in Section 2 – see e.g. [9] and [23].

*Boundary conditions.* Denote by  $H_{\mathbb{C}}^1[0, 1] = H^1([0, 1], \mathbb{C})$  the Sobolev space of functions  $f: [0, 1] \rightarrow \mathbb{C}$  which together with their distributional derivative  $\partial_x f$  are in  $L_{\mathbb{C}}^2[0, 1]$ . On  $H_{\mathbb{C}}^1[0, 1]$  we define the following three boundary conditions (bc),

$$(\text{per } +) \quad f(1) = f(0); \quad (\text{per } -) \quad f(1) = -f(0); \quad (\text{dir}) \quad f(1) = f(0) = 0.$$

The corresponding subspaces of  $H_{\mathbb{C}}^1[0, 1]$  are defined by

$$H_{bc}^1 = \{f \in H_{\mathbb{C}}^1[0, 1] : f \text{ satisfies (bc)}\},$$

and their duals are denoted by  $H_{bc}^{-1} := (H_{bc}^1)'$ . Note that  $H_{\text{per } +}^1$  can be canonically identified with the Sobolev space  $H^1(\mathbb{R}/\mathbb{Z}, \mathbb{C})$  of 1-periodic functions  $f: \mathbb{R} \rightarrow \mathbb{C}$  which together with their distributional derivative are in  $L_{\text{loc}}^2(\mathbb{R}, \mathbb{C})$ . Analogously,  $H_{\text{per } -}^1$  can be identified with the subspace of  $H^1(\mathbb{R}/2\mathbb{Z}, \mathbb{C})$  consisting of functions  $f: \mathbb{R} \rightarrow \mathbb{C}$  with  $f, \partial_x f \in L_{\text{loc}}^2(\mathbb{R}, \mathbb{C})$  satisfying  $f(x+1) = -f(x)$  for all  $x \in \mathbb{R}$ . In the sequel we will not distinguish these pairs of spaces. Furthermore, note that  $H_{\text{dir}}^1$  is a subspace of  $H_{\text{per } +}^1$  as well as of  $H_{\text{per } -}^1$ . Denote by  $\langle \cdot, \cdot \rangle_{bc}$  the extension of the  $L^2$ -inner product  $\langle f, g \rangle_{\mathcal{I}} = \int_0^1 f(x)g(x) \, dx$  to a sesquilinear pairing of  $H_{bc}^{-1}$  and  $H_{bc}^1$ . Finally, we record that the multiplication

$$H_{bc}^1 \times H_{bc}^1 \rightarrow H_{\text{per } +}^1, \quad (f, g) \mapsto fg, \quad (47)$$

and the complex conjugation  $H_{bc}^1 \rightarrow H_{bc}^1, f \mapsto \bar{f}$  are bounded operators.

*Multiplication operators.* For  $q \in H_{\text{per}+}^{-1}$  define the operator  $V_{bc}$  of multiplication by  $q$ ,  $V_{bc}: H_{bc}^1 \rightarrow H_{bc}^{-1}$  as follows: for any  $f \in H_{bc}^1$ ,  $V_{bc}f$  is the element in  $H_{bc}^{-1}$  given by

$$\langle V_{bc}f, g \rangle_{bc} := \langle q, \bar{f}g \rangle_{\text{per}+}, \quad g \in H_{bc}^1.$$

In view of (47),  $V_{bc}$  is a well defined bounded linear operator.

**Lemma C.1** *Let  $q \in H_{\text{per}+}^{-1}$ . For any  $g \in H_{\text{dir}}^1$ , the restriction  $(V_{\text{per} \pm} g)|_{H_{\text{dir}}^1} : H_{\text{dir}}^1 \rightarrow \mathbb{C}$  coincides with  $V_{\text{dir}}g: H_{\text{dir}}^1 \rightarrow \mathbb{C}$ .  $\times$*

*Proof.* Since any  $h \in H_{\text{dir}}^1$  is also in  $H_{\text{per}+}^1$ , the definitions of  $V_{\text{per}+}$  and  $V_{\text{dir}}$  imply

$$\langle V_{\text{per}+}g, h \rangle_{\text{per}+} = \langle q, \bar{g}h \rangle_{\text{per}+} = \langle V_{\text{dir}}g, h \rangle_{\text{dir}},$$

which gives  $(V_{\text{per}+}g)|_{H_{\text{dir}}^1} = V_{\text{dir}}g$ . Similarly, one sees that  $V_{\text{per}-}g|_{H_{\text{dir}}^1} = V_{\text{dir}}g$ .  $\blacksquare$

It is convenient to introduce also the space  $H_{\text{per}+}^1 \oplus H_{\text{per}-}^1$  and define the multiplication operator  $V$  of multiplication by  $q$

$$V: H_{\text{per}+}^1 \oplus H_{\text{per}-}^1 \rightarrow H_{\text{per}+}^{-1} \oplus H_{\text{per}-}^{-1}, \quad (f, g) \mapsto (V_{\text{per}+}f, V_{\text{per}-}g).$$

We note that  $H_{\text{per}+}^1 \oplus H_{\text{per}-}^1$  can be canonically identified with  $H^1(\mathbb{R}/2\mathbb{Z}, \mathbb{C})$ ,

$$H^1(\mathbb{R}/2\mathbb{Z}, \mathbb{C}) \rightarrow H_{\text{per}+}^1 \oplus H_{\text{per}-}^1, \quad f \mapsto (f^+, f^-),$$

where  $f^+(x) = \frac{1}{2}(f(x) + f(x+1))$  and  $f^-(x) = \frac{1}{2}(f(x) - f(x+1))$ . Its dual is denoted by  $H^{-1}(\mathbb{R}/2\mathbb{Z}, \mathbb{C})$ .

*Fourier basis.* The spaces  $H_{\text{per} \pm}^1$ ,  $H^1(\mathbb{R}/2\mathbb{Z}, \mathbb{C})$  and  $H_{\text{dir}}^1$  and their duals admit the following standard Fourier basis. Recall from Appendix A that for any  $s \in \mathbb{R}$  and  $1 \leq p \leq \infty$ , we denote by  $\ell_{\mathbb{C}}^{s,p} \equiv \ell^{s,p}(\mathbb{Z}, \mathbb{C})$  the sequence space

$$\ell_{\mathbb{C}}^{s,p} = \{z = (z_k)_{k \in \mathbb{Z}} \subset \mathbb{C} : \|z\|_{s,p} < \infty\}.$$

Basis for  $H_{\text{per}+}^1, H_{\text{per}+}^{-1}$ . Any element  $f \in H_{\text{per}+}^1$   $[H_{\text{per}+}^{-1}]$  can be represented as  $f = \sum_{m \in \mathbb{Z}} f_m e_m$ ,  $e_m(x) := e^{im\pi x}$ , where  $(f_m)_{m \in \mathbb{Z}} \in h_{\mathbb{C}}^1$   $[h_{\mathbb{C}}^{-1}]$  and

$$f_{2m} = \langle f, e_{2m} \rangle_{\text{per}+}, \quad f_{2m+1} = 0, \quad \forall m \in \mathbb{Z}.$$

Furthermore, for any  $q = \sum_{m \in \mathbb{Z}} q_m e_m \in H_{\text{per}+}^{-1}$ ,

$$V_{\text{per}+}f = \sum_{n \in \mathbb{Z}} \left( \sum_{m \in \mathbb{Z}} q_{n-m} f_m \right) e_n \in H_{\text{per}+}^{-1}.$$

Note that by Lemma A.3,  $(\sum_{m \in \mathbb{Z}} q_{n-m} f_m)_{n \in \mathbb{Z}}$  is in  $h_{\mathbb{C}}^{-1}$ .

Basis for  $H_{\text{per}-}^1, H_{\text{per}-}^{-1}$ . Any element  $f \in H_{\text{per}-}^1 [H_{\text{per}-}^{-1}]$  can be represented as  $f = \sum_{m \in \mathbb{Z}} f_m e_m$  where  $(f_m)_{m \in \mathbb{Z}} \in h_{\mathbb{C}}^1 [h_{\mathbb{C}}^{-1}]$  and

$$f_{2m+1} = \langle f, e_{2m+1} \rangle_{\text{per}-}, \quad f_{2m} = 0, \quad \forall m \in \mathbb{Z}.$$

Similarly, for any  $q = \sum_{m \in \mathbb{Z}} q_m e_m \in H_{\text{per}+}^{-1}$ ,

$$V_{\text{per}-} f = \sum_{n \in \mathbb{Z}} \left( \sum_{m \in \mathbb{Z}} q_{n-m} f_m \right) e_n \in H_{\text{per}-}^{-1}.$$

Basis for  $H^1(\mathbb{R}/2\mathbb{Z}, \mathbb{C}), H^{-1}(\mathbb{R}/2\mathbb{Z}, \mathbb{C})$ . Any element  $f \in H^1(\mathbb{R}/2\mathbb{Z}, \mathbb{C}) [H^{-1}(\mathbb{R}/2\mathbb{Z}, \mathbb{C})]$  can be represented as  $f = \sum_{m \in \mathbb{Z}} f_m e_m$  where  $f_m = \langle f, e_m \rangle$ . Here  $\langle f, g \rangle := \frac{1}{2} \int_0^2 f(x) \overline{g(x)} dx$  denotes the normalized  $L^2$ -inner product on  $[0, 2]$  extended to a sesquilinear pairing between  $H^1(\mathbb{R}/2\mathbb{Z}, \mathbb{R})$  and its dual. In particular, for  $f \in H^1(\mathbb{R}/2\mathbb{Z}, \mathbb{C})$ , and  $m \in \mathbb{Z}$ ,

$$\langle f, e_m \rangle = \frac{1}{2} \int_0^2 f(x) e^{-im\pi x} dx.$$

For any  $q = \sum_{m \in \mathbb{Z}} q_m e_m \in H_{\text{per}+}^{-1} \hookrightarrow H^1(\mathbb{R}/2\mathbb{Z}, \mathbb{C})$

$$Vf = \sum_{n \in \mathbb{Z}} \left( \sum_{m \in \mathbb{Z}} q_{n-m} f_m \right) e_n \in H^{-1}(\mathbb{R}/2\mathbb{Z}, \mathbb{C}).$$

Basis for  $H_{\text{dir}}^1, H_{\text{dir}}^{-1}$ . Note that  $(\sqrt{2} \sin(m\pi x))_{m \geq 1}$  is an  $L^2$ -orthonormal basis of  $L_0^2([0, 1], \mathbb{C})$ . Hence any element  $f \in H_{\text{dir}}^1$  can be represented as

$$f(x) = \sum_{m \geq 1} \langle f, s_m \rangle_{\mathcal{I}} s_m(x) = \frac{1}{2} \sum_{m \in \mathbb{Z}} f_m^{\sin} s_m(x), \quad s_m(x) = \sqrt{2} \sin(m\pi x),$$

where  $f_m^{\sin} = \int_0^1 f(x) s_m(x) dx$ ,  $m \in \mathbb{Z}$ . For any element  $g \in H_{\text{dir}}^{-1}$  one gets by duality

$$g = \frac{1}{2} \sum_{m \in \mathbb{Z}} g_m^{\sin} s_m, \quad g_m^{\sin} = \langle g, s_m \rangle_{\text{dir}}.$$

One verifies that  $g_{-m}^{\sin} = -g_m^{\sin}$  for all  $m \in \mathbb{Z}$  and  $\sum_{m \in \mathbb{Z}} \langle m \rangle^{-2} |g_m^{\sin}|^2 < \infty$ . For any  $q \in H_{0,\mathbb{C}}^{-1}$  with  $\|q\|_{t,2} < \infty$  and  $-1 < t < -1/2$ , we need to expand for a given  $f \in H_{\text{dir}}^1$ ,  $V_{\text{dir}} f \in H_{\text{dir}}^{-1}$  in its sine series  $\frac{1}{2} \sum_{m \in \mathbb{Z}} (V_{\text{dir}} f)_m^{\sin} s_m$  where by the definition of  $V_{\text{dir}}$

$$(V_{\text{dir}} f)_m^{\sin} = \langle V_{\text{dir}} f, s_m \rangle_{\text{dir}} = \langle q, \bar{f} s_m \rangle_{\text{per}+} = \frac{1}{2} \sum_{n \in \mathbb{Z}} f_n^{\sin} \langle q, s_n s_m \rangle_{\text{per}+}.$$

Using that  $f_{-n}^{\sin} = -f_n^{\sin}$  for any  $n \in \mathbb{Z}$  and

$$s_m(x) s_n(x) = \cos((m-n)\pi x) - \cos((m+n)\pi x)$$



it follows that for any  $m \in \mathbb{Z}$

$$\frac{1}{2} \sum_{\substack{m-n \text{ even} \\ n \in \mathbb{Z}}} f_n^{\sin} \langle q, s_n s_m \rangle_{\text{per}+} = \sum_{\substack{m-n \text{ even} \\ n \in \mathbb{Z}}} f_n^{\sin} \langle q, \cos((m-n)\pi x) \rangle_{\text{per}+}.$$

Note that  $\langle q, \cos((m-n)\pi x) \rangle_{\text{per}+}$  is well defined as  $\cos((m-n)\pi x) \in H_{\text{per}+}^1$  if  $m-n$  is even. If  $m-n$  is odd, we decompose the difference of the cosines in  $H_{\text{per}+}^1$  as follows

$$\begin{aligned} & \cos((m-n)\pi x) - \cos((m+n)\pi x) \\ &= (\cos((m-n)\pi x) - \cos(\pi x)) - (\cos((m+n)\pi x) - \cos(\pi x)) \end{aligned}$$

and then obtain, using again that  $f_{-n}^{\sin} = -f_n^{\sin}$  for all  $n \in \mathbb{Z}$ ,

$$\frac{1}{2} \sum_{\substack{m-n \text{ odd} \\ n \in \mathbb{Z}}} f_n^{\sin} \langle q, s_n s_m \rangle_{\text{per}+} = \sum_{\substack{m-n \text{ odd} \\ n \in \mathbb{Z}}} f_n^{\sin} \langle q, \cos((m-n)\pi x) - \cos(\pi x) \rangle_{\text{per}+}.$$

Altogether we have shown that

$$V_{\text{dir}} f = \frac{1}{2} \sum_{m \in \mathbb{Z}} \left( \sum_{n \in \mathbb{Z}} q_{m-n}^{\cos} f_n^{\sin} \right) s_m,$$

where

$$q_k^{\cos} = \begin{cases} \langle q, \cos(k\pi x) \rangle_{\text{per}+}, & \text{if } k \in \mathbb{Z} \text{ even,} \\ \langle q, \cos(k\pi x) - \cos(\pi x) \rangle_{\text{per}+}, & \text{if } k \in \mathbb{Z} \text{ odd.} \end{cases} \quad (48)$$

Since by assumption  $\|q\|_{t,2} < \infty$  with  $-1 < t < -1/2$ , one argues as in [9, Proposition 3.4], using duality and interpolation, that

$$\left( \sum_{m \in \mathbb{Z}} \langle m \rangle^{2t} |q_m^{\cos}|^2 \right)^{1/2} \leq C_t \|q\|_{t,2}. \quad (49)$$

*Schrödinger operators with singular potentials.* For any  $q \in H_{0,\mathbb{C}}^{-1}$  denote by  $L(q)$  the unbounded operator  $-\partial_x^2 + V$  acting on  $H^{-1}(\mathbb{R}/2\mathbb{Z}, \mathbb{C})$  with domain  $H^1(\mathbb{R}/2\mathbb{Z}, \mathbb{C})$ . As  $H^1(\mathbb{R}/2\mathbb{Z}, \mathbb{C}) = H_{\text{per}+}^1 \oplus H_{\text{per}-}^1$  and  $V = V_{\text{per}+} \oplus V_{\text{per}-}$  the operator  $L(q)$  leaves the spaces  $H_{\text{per}\pm}^1$  invariant and  $L(q) = L_{\text{per}+}(q) \oplus L_{\text{per}-}(q)$  with  $L_{\text{per}\pm}(q) = -\partial_x^2 + V_{\text{per}\pm}$ . Hence the spectrum  $\text{spec}(L(q))$  of  $L(q)$ , also referred to as spectrum of  $q$ , is the union  $\text{spec}(L_{\text{per}+}(q)) \cup \text{spec}(L_{\text{per}-}(q))$  of the spectra  $\text{spec}(L_{\text{per}\pm}(q))$  of  $L_{\text{per}\pm}(q)$ . The spectrum  $\text{spec}(L(q))$  is known to be discrete and to consist of complex eigenvalues which, when counted with multiplicities and ordered lexicographically, satisfy

$$\lambda_0^+ \preccurlyeq \lambda_1^- \preccurlyeq \lambda_1^+ \preccurlyeq \dots, \quad \lambda_n^\pm = n^2 \pi^2 + n \ell_n^2,$$

– see e.g. [16]. For any  $q \in H_{0,\mathbb{C}}^{-1}$  there exists  $N \geq 1$  so that

$$|\lambda_n^\pm - n^2 \pi^2| \leq n/2, \quad n \geq N, \quad |\lambda_n^\pm| \leq (N-1)^2 \pi^2 + N/2, \quad n < N, \quad (50)$$

where  $N$  can be chosen locally uniformly in  $q$  on  $H_{0,\mathbb{C}}^{-1}$ . Since for  $q = 0$  and  $n \geq 0$ ,  $\Delta(\lambda_{2n}^+(0), 0) = 2$  and  $\Delta(\lambda_{2n+1}^+(0), 0) = -2$ , all  $\lambda_{2n}^+(0)$  are 1-periodic and all  $\lambda_{2n+1}^+(0)$  are 1-antiperiodic eigenvalues of  $q = 0$ . By considering the compact interval  $[0, q] = \{tq : 0 \leq t \leq 1\} \subset H_{0,\mathbb{C}}^{-1}$  it then follows after increasing  $N$ , if necessary, that for any  $n \geq N$

$$\lambda_n^+(q), \lambda_n^-(q) \in \text{spec}(L_{\text{per}+}(q)), [\text{spec}(L_{\text{per}-}(q))] \quad \text{if } n \text{ even [odd]}. \quad (51)$$

For any  $q \in H_{0,\mathbb{C}}^{-1}$  and  $n \geq N$  the following Riesz projectors are thus well defined on  $H^{-1}(\mathbb{R}/2\mathbb{Z}, \mathbb{C})$

$$P_{n,q} := \frac{1}{2\pi i} \int_{|\lambda - n^2\pi^2| = n} (\lambda - L(q))^{-1} d\lambda.$$

For  $n \geq N$  even [odd], the range of  $P_{n,q}$  is contained in  $H_{\text{per}+}^1$  [ $H_{\text{per}-}^1$ ]. For  $q = 0$ ,  $P_{n,0}$  coincides with the projector  $P_n$  introduced in (7).

Similarly, for any  $q \in H_{0,\mathbb{C}}^{-1}$  denote by  $L_{\text{dir}}(q)$  the unbounded operator  $-\partial_x^2 + V_{\text{dir}}$  acting on  $H_{\text{dir}}^{-1}$  with domain  $H_{\text{dir}}^1$ . Its spectrum  $\text{spec}(L_{\text{dir}}(q))$  is known to be discrete and to consist of complex eigenvalues which, when counted with multiplicities and ordered lexicographically, satisfy

$$\mu_1 \preceq \mu_2 \preceq \dots, \quad \mu_n = n^2\pi^2 + n\ell_n^2,$$

– see e.g. [16]. By increasing the number  $N$  chosen above, if necessary, we can thus assume that

$$|\mu_n - n^2\pi^2| < n/2, \quad n \geq N, \quad |\mu_n| \leq (N-1)^2\pi^2 + N/2, \quad n \leq N. \quad (52)$$

In particular, for any  $n \geq N$ ,  $\mu_n$  is simple and the corresponding Riesz projector

$$\Pi_{n,q} := \frac{1}{2\pi i} \int_{|\lambda - n^2\pi^2| = n} (\lambda - L_{\text{dir}}(q))^{-1} d\lambda$$

is well defined on  $H_{\text{dir}}^{-1}$ . If  $q = 0$ , we write  $\Pi_n$  for  $\Pi_{n,0}$ .

*Regularity of solutions.* In Section 2 we consider solutions  $f$  of the equation  $(L(q) - \lambda)f = g$  in  $\mathcal{F}_{\star,\mathbb{C}}^{s,\infty}$  and need to know their regularity.

**Lemma C.2** *For any  $q \in \mathcal{F}_{0,\mathbb{C}}^{s,\infty}$  with  $-1/2 < s \leq 0$ , the following holds: For any  $g \in \mathcal{F}_{\star,\mathbb{C}}^{s,\infty}$  and any  $\lambda \in \mathbb{C}$ , a solution  $f \in H^1(\mathbb{R}/2\mathbb{Z}, \mathbb{C})$  of the inhomogeneous equation  $(L(q) - \lambda)f = g$  is an element in  $\mathcal{F}_{\star,\mathbb{C}}^{s+2,\infty}$ .  $\times$*

*Proof.* Let  $g \in \mathcal{F}_{\star,\mathbb{C}}^{s,\infty}$  and assume that  $f \in H^1(\mathbb{R}/2\mathbb{Z}, \mathbb{C})$  solves  $(L(q) - \lambda)f = g$ . Write  $(L - \lambda)f = g$  as  $A_\lambda f = Vf - g$  where  $A_\lambda = \partial_x^2 + \lambda$ . Since  $q \in \mathcal{F}_{0,\mathbb{C}}^{s,\infty}$ , Lemma A.2 implies that  $q$  and  $g$  are in  $\mathcal{F}_{\star,\mathbb{C}}^{r,2}$  with  $r = s - 1/2 - \varepsilon$  where  $\varepsilon > 0$  is chosen such that  $r > -1$ . By Lemma A.3 (i),  $Vf \in \mathcal{F}_{\star,\mathbb{C}}^{r,2}$  and hence  $A_\lambda f = Vf - g \in \mathcal{F}_{\star,\mathbb{C}}^{r,2}$  implying that  $f \in \mathcal{F}_{\star,\mathbb{C}}^{r+2,2}$ . Since  $-s - 3/2 \leq -1 < r \leq 0$ , Lemma A.3 (ii) applies. Therefore,  $Vf \in \mathcal{F}_{\star,\mathbb{C}}^{s,\infty}$  and using the equation  $A_\lambda f = Vf - g$  once more one gets  $f \in \mathcal{F}_{\star,\mathbb{C}}^{s+2,\infty}$  as claimed.  $\blacksquare$

For any  $q \in \mathcal{F}_{0,\mathbb{C}}^{s,\infty}$  with  $-1/2 < s \leq 0$ , and  $n \geq n_s$  as in Corollary 2.5, introduce

$$E_n \equiv E_n(q) := \begin{cases} \text{Null}(L(q) - \lambda_n^+) \oplus \text{Null}(L(q) - \lambda_n^-), & \lambda_n^+ \neq \lambda_n^-, \\ \text{Null}(L(q) - \lambda_n^+)^2, & \lambda_n^+ = \lambda_n^-. \end{cases}$$

Then  $E_n$  is a two-dimensional subspace of  $H_{\text{per}+}^1 [H_{\text{per}-}^1]$  if  $n$  is even [odd]. The following result shows that elements in  $E_n$  are more regular.

**Lemma C.3** *For any  $q \in \mathcal{F}_{0,\mathbb{C}}^{s,\infty}$  with  $-1/2 < s \leq 0$  and for any  $n \geq n_s$   $E_n(q) \subset \mathcal{F}_{*,\mathbb{C}}^{s+2,\infty} \cap H_{\text{per}+}^1 [\mathcal{F}_{*,\mathbb{C}}^{s+2,\infty} \cap H_{\text{per}-}^1]$  if  $n$  is even [odd].*  $\times$

*Proof.* By Lemma C.2 with  $g = 0$ , any eigenfunction  $f$  of an eigenvalue  $\lambda$  of  $L(q)$  is in  $\mathcal{F}_{*,\mathbb{C}}^{s+2,\infty}$ . Hence if  $\lambda_n^+ \neq \lambda_n^-$  or if  $\lambda_n^+ = \lambda_n^-$  and has geometric multiplicity two, then  $E_n \subset \mathcal{F}_{*,\mathbb{C}}^{s+2,\infty}$ . Finally, if  $\lambda_n^+ = \lambda_n^-$  is a double eigenvalue of geometric multiplicity 1 and  $g$  is an eigenfunction corresponding to  $\lambda_n^+$ , there exists an element  $f \in H^1(\mathbb{R}/2\mathbb{Z}, \mathbb{C})$  so that  $(L - \lambda_n^+)f = g$ . Since  $g$  is an eigenfunction it is in  $\mathcal{F}_{*,\mathbb{C}}^{s+2,\infty}$  by Lemma C.2 and by applying this lemma once more, it follows that  $f \in \mathcal{F}_{*,\mathbb{C}}^{s+2,\infty}$ . Clearly,  $E_n = \text{span}(g, f)$  and hence  $E_n \subset \mathcal{F}_{*,\mathbb{C}}^{s+2,\infty}$  also in this case. By (51),  $\lambda_n^\pm$  are 1-periodic [1-antiperiodic] eigenvalues of  $q$  if  $n$  is even [odd]. Hence  $E_n \subset \mathcal{F}_{*,\mathbb{C}}^{s+2,\infty} \cap H_{\text{per}+}^1 [\mathcal{F}_{*,\mathbb{C}}^{s+2,\infty} \cap H_{\text{per}-}^1]$  if  $n$  is even [odd] as claimed.  $\blacksquare$

*Estimates for projectors.* The projectors  $P_{n,q} [\Pi_{n,q}]$  with  $n \geq N$  and  $N$  given by (50)-(52) are defined on  $H^{-1}(\mathbb{R}/2\mathbb{Z}, \mathbb{C}) [H_{\text{dir}}^{-1}]$  and have range in  $H^1(\mathbb{R}/2\mathbb{Z}, \mathbb{C}) [H_{\text{dir}}^1]$ . The following result concerns estimates for the restriction of  $P_{n,q} [\Pi_{n,q}]$  to  $L^2 = L^2([0, 2], \mathbb{C}) [L^2(\mathcal{I}) = L^2([0, 1], \mathbb{C})]$  needed in Section 2 for getting the asymptotics of  $\mu_n - \tau_n$  stated in Theorem 2.1 (ii).

**Lemma C.4** *Assume that  $q \in H_{0,\mathbb{C}}^{-1}$  with  $\|q\|_{t,2} < \infty$  and  $-1 < t < -1/2$ . Then there exist constants  $C_t > 0$  (only depending on  $t$ ) and  $N' \geq N$  (with  $N$  as above) so that for any  $n \geq N$  the following holds*

$$(i) \quad \|P_{n,q} - P_n\|_{L^2 \rightarrow L^\infty} \leq C_t \frac{(\log n)^2}{n^{1-|t|}} \|q\|_{t,2}$$

$$(ii) \quad \|\Pi_{n,q} - \Pi_n\|_{L^2(\mathcal{I}) \rightarrow L^\infty(\mathcal{I})} \leq C_t \frac{(\log n)^2}{n^{1-|t|}} \|q\|_{t,2}$$

*The constant  $N'$  can be chosen locally uniformly in  $q$ .*  $\times$

*Proof.* [7, Lemma 25].  $\blacksquare$

**Remark C.5.** We will apply Lemma C.4 for potentials  $q \in \mathcal{F}_{0,\mathbb{C}}^{s,\infty}$  with  $-1/2 < s \leq 0$  using the fact that by the Sobolev embedding theorem there exists  $-1 < t < -1/2$  so that  $\mathcal{F}_{0,\mathbb{C}}^{s,\infty} \hookrightarrow H_{0,\mathbb{C}}^t$ .  $\dashv$

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